Mathematical Discoveries of the Bernoulli Brothers

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This Swiss family produced eight mathematicians in three generations.

We will focus on some of the mathematical discoveries of Jakob I and his brother Johann I.
Nikolaus Bernoulli wanted Jakob to be a Protestant pastor and Johann to be a doctor.

They obeyed their father and earned degrees in theology and medicine, respectively.

But...
Jakob and Johann taught themselves the “new math” – calculus – from Leibniz’s notes and papers.

They started to have contact with Leibniz, and are now known as his most important students.
Jakob Bernoulli (1654-1705)

- learned about mathematics and astronomy
- studied Descarte’s *La Géometrie*, John Wallis’s *Arithmetica Infinitorum*, and Isaac Barrow’s *Lectiones Geometricae*
- convinced Leibniz to change the name of the new math from calculus summatorius to calculus integralis

http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Bernoulli_Jakob.html
Johann Bernoulli (1667-1748)

- studied mathematics and physics
- gave calculus lessons to Marquis de L’Hôpital
- Johann’s greatest student was Euler
- won the Paris Academy’s biennial prize competition three times – 1727, 1730, and 1734

http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Bernoulli_Johann.html
Jakob vs. Johann

- Johann Bernoulli had greater intuitive power and descriptive ability.
- Jakob had a deeper intellect but took longer to arrive at a solution.
Famous Problems

- the catenary (hanging chain)
- the brachistocron (shortest time)
- the divergence of the harmonic series (1/n)
The Catenary: Hanging Chain

- Jakob Bernoulli proposed this problem in the May 1690 edition of *Acta Eruditorum*.

- “And now let this problem be proposed: To find the curve assumed by a loose string hung freely from two fixed points”

- Galileo guessed that this curve was a parabola, but he never proved it.

- Dutch scientist Christiaan Huygens proved in 1646 that this shape is not a parabola, but could not prove what it was.
Solutions for the Catenary

- Huygens, Leibniz, and Johann Bernoulli submitted correct solutions to the catenary.
- Jakob Bernoulli was angry that his younger brother solved this problem and he couldn’t.
- But Jakob later solved general forms of the catenary which allowed for variations, such as an elastic chain or a chain of variable density.
The Catenary Equation

- part of the hyperbolic cosine function
  \[ y = \frac{(e^x + e^{-x})}{2} \]
- During the Seventeenth Century, the number \( \infty \) did not have a symbol.
- The modern equation was introduced in 1757 by the Italian Jesuit Vincenzo Riccati.
Applications of the Catenary

- the Gateway Arch in St. Louis, MO – an inverted catenary.
- suspension bridges, such as the Golden Gate Bridge
- high voltage transmission lines, telephone lines

URL: http://www.stlouisarch.com/photos/riverfront.jpg
The Brachistocrono Problem

- Johann Bernoulli proposed this problem in *Acta Eruditorum* (June, 1696).
- “Suppose two nails are driven at random into a wall, and let the upper nail be connected to the lower by a wire in the shape of a smooth curve. What is the shape of the wire down which a bead will slide so as to pass from the upper nail to the lower in the least possible time?”
- Galileo believed that the solution was the arc of a circle.
- Bernoulli gave mathematicians until January of 1697 to provide a solution.
But only Leibniz solved it by that time, so Johann expanded the time and reworded the problem to avoid any confusion.

“Among the infinitely many curves which join two given points...choose one such that, if the curve is replaced by a thin tube or groove, and a small sphere is placed in it and released, then this will pass from one point to the other in the shortest time.”
Brachistocrono Solution

- Newton, Leibniz, L’Hôpital, and Jakob and Johann Bernoulli provided correct solutions for this problem.

- Johann’s approach used Fermat’s Principle and physics.

- Jakob took a more mathematical approach and arrived at a differential equation.
What Does It Look Like?

- the curve traced by a point on the rim of a wheel as it rolls along a horizontal surface
Brachistocrone Solution, cont.

- upside-down cycloid
- parametric equation
  \[ x = t - \sin(t) \]
  \[ y = 1 - \cos(t) \]

The cycloid had been investigated by so many mathematicians that there seemed to be nothing more to be discovered about it.
The Harmonic Series

- $1/n$
- The terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4},$ etc.
- For a series to converge, the sum of the terms must be approaching one number.

- $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$
- Johann Bernoulli suggested that even though the terms themselves are getting closer to zero, the sum is growing to infinity (sum becomes infinity).
Let’s See It In Action

\[ a_n = \frac{1}{n} \]

Terms

\[ S_n = \sum_{i=1}^{N} a_i \]

Partial Sums

\[ n = 1 \]

www.math.odu.edu.chiicalcanim/series1.avi
“Pathological Counterexample”

- must sum up first 83 terms to get sum > 5.00
- must look at the first 227 terms to get sum > 6.00
- must sum the first 12,368 terms to get sum > 10.00
- Are we sure it doesn’t converge somewhere?
The Divergence of the Harmonic Series

- proof was derived by Johann Bernoulli, but was printed in Jakob Bernoulli’s *Tractatus de seriebus infinitus* (*Treatise on Infinite Series*)

- proof rests on Leibniz’s summation of the convergent series

- **Theorem**: The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} + \ldots$ is infinite.
The Proof

**Proof:** introduce $A = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} + \ldots$, which is the harmonic series lacking the first term.

- change the numerators to 1, 2, 3, etc. so that $A = \frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \ldots$
- We will come back to $A$ later in the proof – so don’t forget about it.
The Proof, cont.

- \( C = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = 1 \)
- \( D = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = C - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} \)
- \( E = \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = D - \frac{1}{6} = 1 - \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \)
- \( F = \frac{1}{20} + \frac{1}{30} + \ldots = E - \frac{1}{12} = \frac{1}{3} - \frac{1}{12} = \frac{1}{4} \)
- \( G = \frac{1}{30} + \ldots = F - \frac{1}{20} = \frac{1}{4} - \frac{1}{20} = \frac{1}{5} \)
- continue...
- now add the left sides of \( C + D + E + F + G \ldots = \frac{1}{2} + \left( \frac{1}{6} + \frac{1}{6} \right) + \left( \frac{1}{12} + \frac{1}{12} + \frac{1}{12} \right) + \left( \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} \right) + \left( \frac{1}{30} + \frac{1}{30} + \frac{1}{30} + \frac{1}{30} + \frac{1}{30} + \ldots \right) \)
- \( \frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \ldots = A \Rightarrow \) previously defined
The Proof, cont.

- But adding the right sides of the equations, 
  \[ C + D + E + F + \ldots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = 1+A \]
- Since \( C+D+E+F+G+\ldots \) equals both \( A \) and \( 1+A \), Johann could only conclude that \( A=1+A \).
- “the whole equals the part”
- To Bernoulli, it could only mean that \( 1+A \) is an infinite quantity. Thus his argument was complete.
Modern Critics of the Proof

- Bernoulli treated infinite series as individual terms to be manipulated at will.
- Today, much more care is taken when working with series.
- Bernoulli proved divergence by proving that $A=1+A$.
- Today, one would fix a number $N$ and show that the series exceeds that number $N$. 
In Bernoulli’s Defense...

- Johann Bernoulli wrote this proof approximately 150 years before a truly rigorous theory of series was developed.
- One cannot deny the mathematical insight and cleverness of Bernoulli’s argument.
Other Bernoulli Contributions

- L’Hôpital’s Rule
- convergence of the sum of the reciprocals of the squares \(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\)
- *Ars Conjectandi* (*The Art of Conjecturing*)
- Bernoulli Theorem: Law of Large Numbers
- reflection and refraction
- analytical trigonometry
- early use of polar coordinates
The Bernoulli brothers made many contributions to mathematics during their lives.

Their mathematical skills were on the cutting edge of ‘new math’ during the Seventeenth and Eighteenth Centuries.

Where would we be mathematically without their work?
References