Difference Equations
Introduction

1. Brief overview of the basics of difference equations

2. Study of a general epidemic model that utilizes generating functions

3. Research and findings of Union University influenza data using a difference equation model
What is a Difference Equation?

At its basics, a difference equation is a relation between consecutive elements of a sequence. Difference equations enable the mathematician to use a given set of values in order to determine the value of a function.
An example would be in finding the value of an investment:

Let $y(t)$ be the value of an investment after $t$ quarters of a year. Our original investment will be $24$, so $y(0) = 24$. If our interest rate is $1.75\%$ per quarter, then $y(t)$ satisfies the difference equation

$$y(t + 1) = y(t) + 0.0175y(t)$$
$$= y(t)[1 + 0.0175]$$
$$= (1.0175)y(t)$$

for $t = 0, 1, 2, \ldots$. Computing $y$ recursively, we have

$$y(1) = 24 \cdot 1.0175$$
$$y(2) = 24 \cdot (1.0175)^2$$
$$\vdots$$

$$y(t) = 24 \cdot (1.0175)^t.$$
Difference Operator

Let \( y(t) \) be a function of a real or complex variable \( t \). The "difference operator" \( \Delta \) is defined by

\[
\Delta y(t) = y(t+1) - y(t).
\]
There are several rules, theorems, and properties that go along with the difference operator.

For instance:

- the higher order difference rule (similar to a 2nd derivative in differential calculus),

- the factorial function (a version of the power rule for solving finite differences)
To show how simple and accessible a proof in difference calculus can be, an example is given for the theorem

Theorem: $\Delta a^t = (a - 1)a^t$.

Proof: $\Delta a^t = a^{t+1} - a^t = a^t(a^1 - 1)$. 
Summation

An “indefinite sum” of $y(t)$, denoted $\sum y(t)$ is any function so that

$$\Delta(\sum y(t)) = y(t)$$

for all $t$ in the domain of $y$. 
Generating Function

Let \( \{y_k\} \) be a sequence of constants. Suppose there is a function \( g(x) \) so that

\[
g(x) = \sum_{k=0}^{\infty} y_k x^k
\]

for all \( x \) in an open interval about 0. Then \( g \) is called the “generating function” for \( \{y_k\} \).
Finding a Generating Function

Let $y_k = C^k$, for some constant $C$. To compute the generating function for $\{y_k\}$, the series must be summed,

$$\sum_{k=0}^{\infty} C^k x^k = \sum_{k=0}^{\infty} (Cx)^k$$

This can be recognized as a geometric series and the generating function is found.

$$\frac{1}{1-Cx} = g(x)$$

for $|Cx| < 1$. 
Epidemiology Model
The epidemic model investigated below is presented in Kelley and Peterson, pages 87 and 88 and Lauwerier, page 162.

\( x_n \) = the fraction of susceptible individuals in a certain population during the \( n^{th} \) day of an epidemic

\( A_k \) = a measure of how infectious the ill individuals are during the \( k^{th} \) day

\( \epsilon \) = a small positive constant representing the carriers (those who can not get the disease but are able to spread it to others).

\[
\log \frac{1}{x_{n+1}} = \sum_{k=0}^{n} (1 + \epsilon - x_{n-k}) A_k, \quad (n \geq 0)
\]

With substitutions and the utilization of exponent rules and properties of the natural log, this equation can be rewritten:

\[
y_{n+1} = \sum_{k=0}^{n} (\epsilon + y_{n-k}) A_k.
\]

where \( y_n \) is the fraction of people that have the disease.
The method of generating functions can be applied here because of the form of the sum $\sum_{k=0}^{n} y_{n-k} A_k$, which is called a sum of "convolution type."

Now, a generating function $Y(t)$ must be derived for \(\{y_n\}\),

$$Y(t) = \sum_{n=0}^{\infty} y_n t^n,$$

and set

$$A(t) = \sum_{n=0}^{\infty} A_n t^{n+1}.$$

By multiplying the two power series and factoring, the product is

$$A(t)Y(t) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} y_{n-k} A_k \right) t^{n+1}.$$
Returning to the equation $y_{n+1} = \sum_{k=0}^{n} (\epsilon + y_{n-k})A_k$ and distributing $A_k$ as well as multiplying both sides of this difference equation by $t^{n+1}$ and summing results in:

$$\sum_{n=0}^{\infty} y_{n+1} t^{n+1} = \epsilon \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} A_k \right) t^{n+1} + \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} A_k y_{n-k} \right) t^{n+1}.$$

By simplification, substitution, and factoring, a generating function for $\{y_n\}$ is derived:

$$Y(t) = \frac{\epsilon A(t)}{\left(1 - \epsilon - A(t)\right)}.$$
In a few special cases, the sequence \( \{y_n\} \) can be computed explicitly. The specific model that will be used with the Union University data is presented on pages 88 and 89 of the Kelley and Peterson text.

Conditions:

\[
A_k = c \alpha^k, \ 0 < \alpha < 1.
\]

- \( c \) is a constant that represents the infectiousness on day 0.
- \( \alpha \) is the rate at which the infectiousness declines daily.

Set \( A(t) = \frac{ct}{1-\alpha t} \) and \( Y(t) = \frac{e^{ct}}{1-\frac{\alpha}{1-\alpha ct}} \).

By partial fractions, simple algebra, and recognition of geometric series, the equation for \( Y(t) \) can be rewritten,

\[
Y(t) = \left( \frac{c}{1-(\alpha + c)} \right) \left[ \sum_{n=0}^{\infty} \alpha^n t^n - \sum_{n=0}^{\infty} (\alpha + c)^n t^n \right].
\]

So, for a single value of \( y_n \), it is written

\[
y_n = \left( \frac{c}{1-(\alpha + c)} \right)[1 - (\alpha + c)^n].
\]
Union University and Influenza

\texttt{ListPlot[Table[uuprebreak, \{n, 0, 40\}]]}
Graph the generating function with particular values of $\epsilon$, $\alpha$, and $c$.

$\epsilon := .0088$
$\alpha := .53$
$c := .453$

```
ListPlot[
  Table[2857 * ($\epsilon$ * $c$) / (1 - ($\alpha$ + $c$)) (1 - ($\alpha$ + $c$)^n),
    {n, 0, 50}], uuprebreak]
```
Expanding the domain shows when the graph will level off.

```
ListPlot@Table[2857 * (e * c) / (1 - (a + c)) (1 - (a + c) ^ n), {n, 0, 396}],
```

uuprebreak
Now, look at the post-fall break data and compare.

```
uupostbreak := {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 328, 329, 333, 334, 337, 339, 341, 341, 343, 343, 343, 345, 345, 347, 347, 349, 352, 353}
```

```
ListPlot[
{Table[2857 * (e * c) / (1 - (alpha + c)) (1 - (alpha + c) ^ n),
{n, 0, 50}], uuprebreak, uupostbreak}]
```
Possible Conclusions to Research:

Natural trend

Extended break

Need more data
References


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