An AIDS Epidemic Model

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Epidemic - a contagious disease that affects an excessive number of people at a time

Past Epidemics

• “Plague of Justinian” 541 A.D.
• Bubonic Plague 1338
• Influenza 1918
• Polio early 1900’s
Simplest Mathematical Model

- \( t = \) time
- \( R(t) = \) # of infected people at \( t \)
- \( k = \) constant of proportionality

\[
d\frac{R(t)}{dt} = kR(t)
\]

\[
R(t) = R_0 e^{kt}
\]

\( R_0 = \) number of infected people at time zero
Logistic Model

- **t = time**
- **R(t) = # of infected people at t**
- **N = total people in population**
- **Susceptible = N - R(t)**

\[
\frac{dR(t)}{dt} = kR(t)(N - R(t))
\]
Formula for Logistic Growth

\[ R(t) = \frac{NR_0}{R_0 + (N - R_0)e^{-kNt}} \]
Two important facts about the behavior of $R(t)$

**Small $t$:**

$$R(t) \approx R_0 e^{kNt}$$

For small $t$, logistic growth looks like exponential growth.

**Large $t$:**

$$\lim_{t \to \infty} R(t) = N$$
The growth of any epidemic at time $t$
Logistic Model
Assumptions

- $R(t)$ is assumed to be a continuous function.
- We assumed that the growth rate is proportional to the product of the numbers of infecteds and susceptibles.
- Infecteds and susceptibles are the only two categories of people.
- A newly infected person automatically develops the epidemic.
- Any person can infect any other person.
Facts about AIDS

- AIDS is the fifth leading cause of death.
- Results from an HIV infection
- Needle-sharing, blood transfusions, and sexual contact
- Sexual contact results in the largest percentage of AIDS cases
- Latency period: 2-18 years
Saturation Wave Model

- **Six Steps**
  - Latency Period
  - Formula for Derivative of A(t)
  - Heterogeneous Behavior
  - Growth in Single Risk Group
  - Saturation Wave and HIV Infection
  - Cubic Growth of AIDS
Step 1: Latency Period

- \( L(t) = \) probability density function for the latency period

\[
\int_{\tau}^{\tau + \Delta \tau} L(t) \, dt
\]
Step 2

\[ A(t + \Delta \tau) - A(t) \approx \sum_{i=1}^{n} \left[ H(t - \tau_i) - H(t - \tau_i - \Delta \tau) \right] L(\tau_i) \Delta \tau \]

\[ A'(t) = \int_{-\infty}^{t} H'(t - \tau) L(\tau) d\tau \]
Step 3: Heterogeneous Behavior

- $r =$ risk factor
- $N(r) =$ # of individuals with risk, $r$

\[ N(r) \approx \frac{N_0}{r^3} \]
Step 4: Growth in single risk group

\[ \lambda = \text{Proportionality constant} \]

\[ H_r(t) = \# \text{ of individuals with risk } r \text{ that have the HIV infection} \]

\[ H_r(t) = H_r(0)e^{\lambda rt} \]

\[ H_r(0) = \# \text{ of infected individuals when we start measuring time} \]
When will the entire group be infected?

\[ N(r) = H_r (0)e^{\lambda rt} \]

At what time will this occur?

\[ t = \frac{1}{\lambda r} \ln \left( \frac{N(r)}{H_r (0)} \right) \]
Step 5: Saturation Wave and HIV Infection

\[ \int_{r_*}^{\infty} N(r) \, dr = \int_{r_*}^{\infty} \frac{N_0}{r^3} \, dr = \frac{N_0}{2r_*^2} \]

\( r_* \) = group that just reached saturation
Step 6: Cubic Growth of AIDS

\[ \Gamma(t) = 0.0625, \quad 5 \leq t \leq 18 \]

\[ A'(t) = 0.0625 \cdot K(t - 2)^2 \]

\[ A(t) = 0.0625K \cdot \frac{1}{3}(t - 2)^3 \]

The cumulative number of AIDS cases is a cubic function of time.
Cubic Growth of AIDS
An Estimate of AIDS

The year 1998: 900,307

The year 1999: 1,068,710
The End