Mathematical Discoveries of the Bernoulli Brothers

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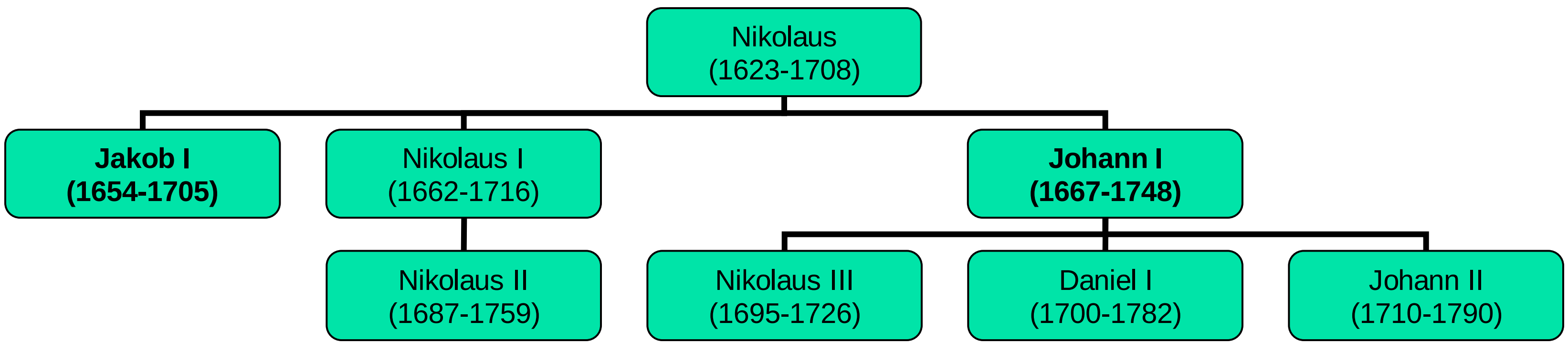
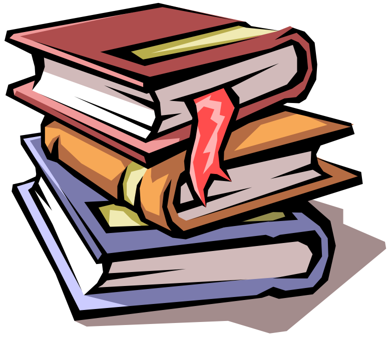
Bernoulli Family Tree



- This Swiss family produced eight mathematicians in three generations.
- We will focus on some of the mathematical discoveries of Jakob I and his brother Johann I.

Some History

- Nikolaus Bernoulli wanted Jakob to be a Protestant pastor and Johann to be a doctor.
- They obeyed their father and earned degrees in theology and medicine, respectively.
- But…



Some History, cont.

http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Leibniz.html

- Jakob and Johann taught themselves the “new math” – calculus – from Leibniz’s notes and papers.
- They started to have contact with Leibniz, and are now known as his most important students.

Jakob Bernoulli (1654-1705)

- learned about mathematics and astronomy
- studied Descarte's *La Géometrie,* John Wallis's *Arithmetica Infinitorum*, and Isaac Barrow's *Lectiones Geometricae*
- convinced Leibniz to change the name of the new math from calculus sunmatorius to calculus integralis

http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Bernoulli_Jakob.html

Johann Bernoulli (1667-1748)

http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Bernoulli_Johann.html

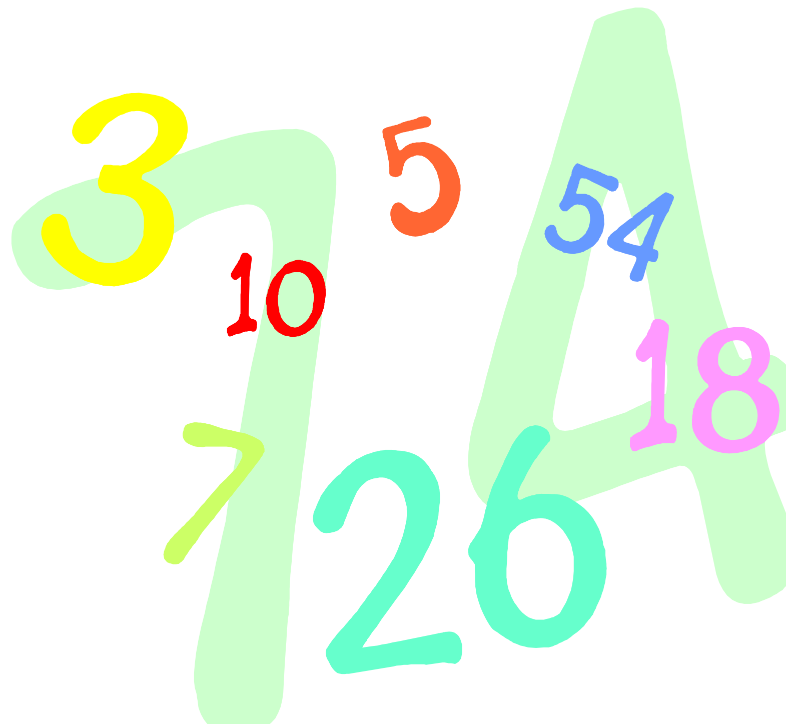
- studied mathematics and physics
- gave calculus lessons to Marquis de L'Hôpital
- Johann's greatest student was Euler
- won the Paris Academy's biennial prize competition three times – 1727, 1730, and 1734

Jakob vs. Johann

- Johann Bernoulli had greater intuitive power and descriptive ability
- Jakob had a deeper intellect but took longer to arrive at a solution

Famous Problems

- the catenary (hanging chain)
- the brachistocrone (shortest time)
- the divergence of the harmonic series (1/n)



The Catenary: Hanging Chain

- Jakob Bernoulli proposed this problem in the May 1690 edition of *Acta Eruditorum*.
- “And now let this problem be proposed: To find the curve assumed by a loose string hung freely from two fixed points”
- Galileo guessed that this curve was a parabola, but he never proved it.
- Dutch scientist Christiaan Huygens proved in 1646 that this shape is not a parabola, but could not prove what it was.

Solutions for the Catenary

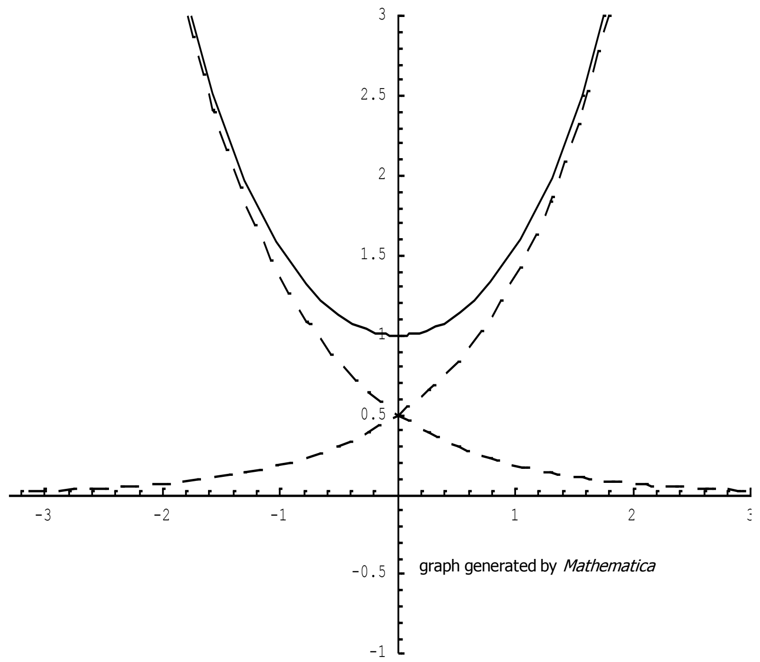
- Huygens, Leibniz, and Johann Bernoulli submitted correct solutions to the catenary.
- Jakob Bernoulli was angry that his younger brother solved this problem and he couldn't.
- But Jakob later solved general forms of the catenary which allowed for variations, such as an elastic chain or a chain of variable density.

The Catenary Equation

- part of the hyperbolic cosine function

  $y = \left(e^{x}+e^{-x}\right)/_{2}$

- During the Seventeenth Century, the number ↖ did not have a symbol.
- The modern equation was introduced in 1757 by the Italian Jesuit Vincenzo Riccati.



Applications of the Catenary

http://www.stlouisarch.com/photos/riverfront.jpg

- the Gateway Arch in St. Louis, MO – an inverted catenary.
- suspension bridges, such as the Golden Gate Bridge
- high voltage transmission lines, telephone lines

The Brachistocrone Problem

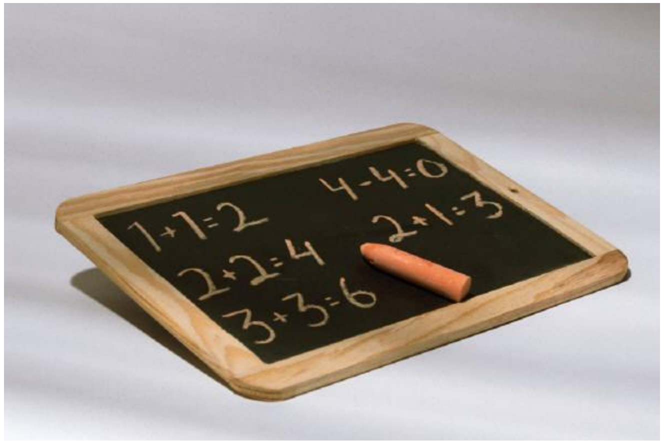
- Johann Bernoulli proposed this problem in *Acta Eruditorum* (June, 1696).
- “Suppose two nails are driven at random into a wall, and let the upper nail be connected to the lower by a wire in the shape of a smooth curve. What is the shape of the wire down which a bead will slide so as to pass from the upper nail to the lower in the least possible time?”
- Galileo believed that the solution was the arc of a circle.
- Bernoulli gave mathematicians until January of 1697 to provide a solution.

Brachistocrone, cont.

- But only Leibniz solved it by that time, so Johann expanded the time and reworded the problem to avoid any confusion.
- “Among the infinitely many curves which join two given points…choose one such that, if the curve is replaced by a thin tube or groove, and a small sphere is placed in it and released, then this will pass from one point to the other in the shortest time.”

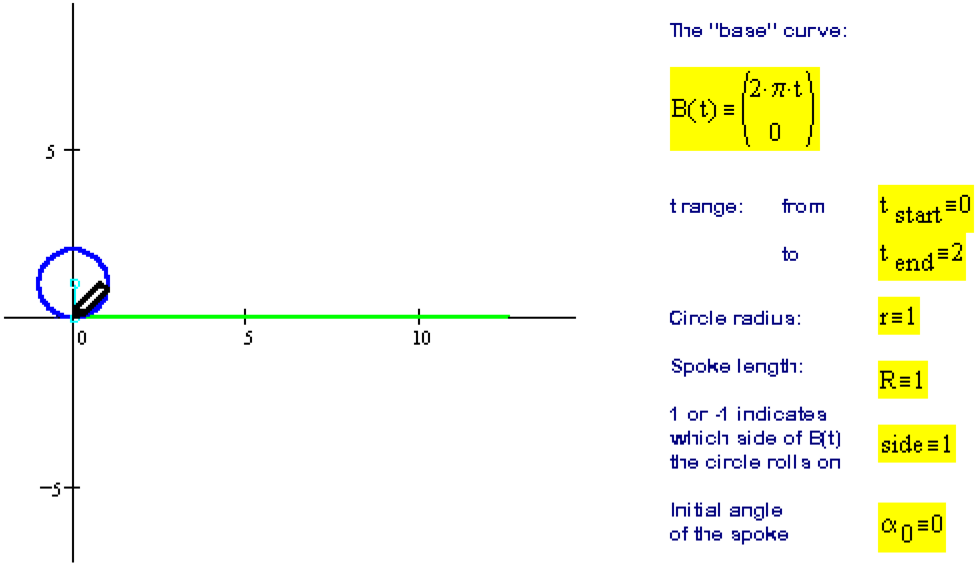
Brachistocrone Solution

- Newton, Leibniz, L'Hôpital, and Jakob and Johann Bernoulli provided correct solutions for this problem.
- Johann's approach used Fermat's Principle and physics.
- Jakob took a more mathematical approach and arrived at a differential equation.



What Does It Look Like?

- the curve traced by a point on the rim of a wheel as it rolls along a horizontal surface



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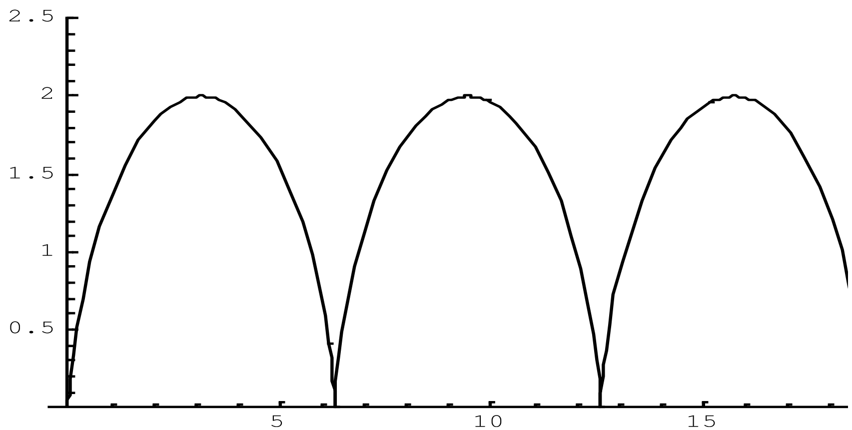
Brachistocrone Solution, cont.

- upside-down cycloid
- parametric equation
  - $x = t - \sin(t)$
    $y = 1 - \cos(t)$



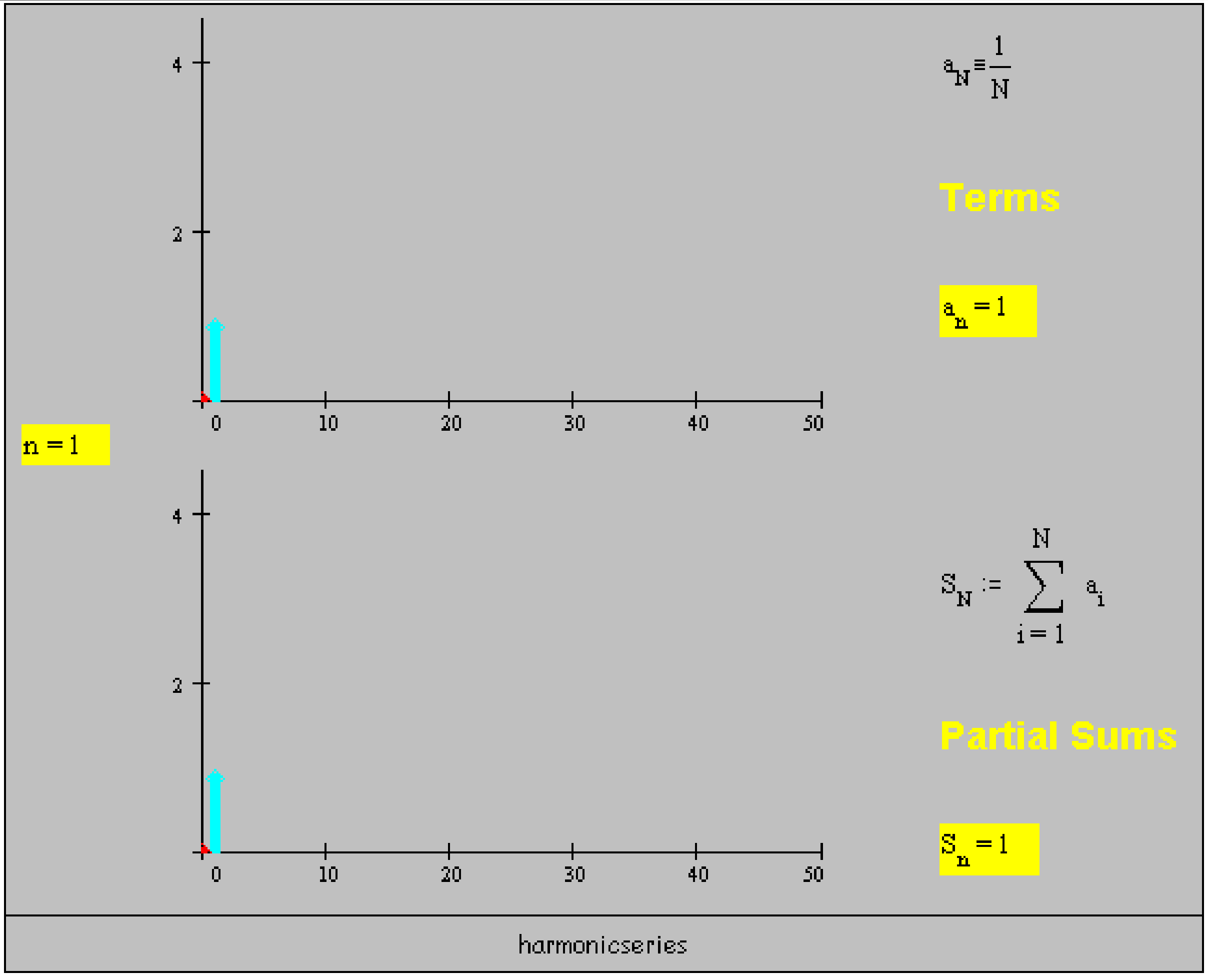
graph generated by *Mathematica*

- The cycloid had been investigated by so many mathematicians that there seemed to be nothing more to be discovered about it.

The Harmonic Series

- $1/n$
- The terms are 1, $^1/_2$, $^1/_3$, $^1/_4$, etc.
- For a series to converge, the <u>sum</u> of the terms must be approaching one number.
- $1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + \ldots$
- Johann Bernoulli suggested that even though the terms themselves are getting closer to zero, the sum is growing to infinity (sum becomes infinity).

Let’s See It In Action



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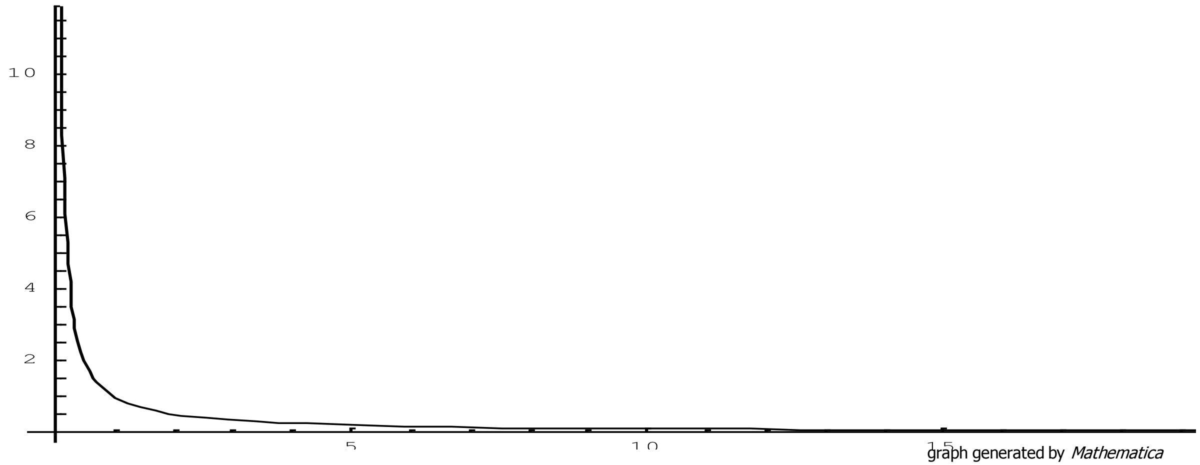
“Pathological Counterexample”



- must sum up first 83 terms to get sum > 5.00
- must look at the first 227 terms to get sum > 6.00
- must sum the first 12,368 terms to get sum > 10.00
- Are we sure it doesn’t converge somewhere?

The Divergence of the Harmonic Series

- proof was derived by Johann Bernoulli, but was printed in Jakob Bernoulli's *Tractatus de seriebus infinitus (Treatise on Infinite Series)*
- proof rests on Leibniz's summation of the convergent series
- <u>Theorem</u>: The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} + \ldots$ is infinite.

The Proof

- Proof: introduce A= $^1/_2 + ^1/_3 + ^1/_4 + \ldots + ^1/_k + \ldots$, which is the harmonic series lacking the first term.
- change the numerators to 1, 2, 3, etc. so that A=$^1/_2 + ^2/_6 + ^3/_{12} + ^4/_{20} + \ldots$
- We will come back to A later in the proof – so don't forget about it.

The Proof, cont.

- $C = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = 1$
- $D = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = C - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$
- $E = \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = D - \frac{1}{6} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$
- $F = \frac{1}{20} + \frac{1}{30} + \ldots = E - \frac{1}{12} = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$
- $G = \frac{1}{30} + \ldots = F - \frac{1}{20} = \frac{1}{4} - \frac{1}{20} = \frac{1}{5}$
- continue…
- now add the left sides of C+D+E+F+G… = $\frac{1}{2} + (\frac{1}{6} + \frac{1}{6}) + (\frac{1}{12} + \frac{1}{12} + \frac{1}{12}) + (\frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20}) + (\frac{1}{30} + \frac{1}{30} + \frac{1}{30} + \frac{1}{30} + \frac{1}{30}) + \ldots$
- $\frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \ldots = A$ → previously defined

The Proof, cont.

- But adding the right sides of the equations, C + D + E + F + … = 1 + $^1/_2$+ $^1/_3$ + $^1/_4$ + $^1/_5$ +… = 1+A
- Since C+D+E+F+G+… equals both A and 1+A, Johann could only conclude that A=1+A.
- “the whole equals the part”
- To Bernoulli, it could only mean that 1+A is an infinite quantity. Thus his argument was complete.

Modern Critics of the Proof

- Bernoulli treated infinite series as individual terms to be manipulated at will.
- Today, much more care is taken when working with series.
- Bernoulli proved divergence by proving that A=1+A.
- Today, one would fix a number *N* and show that the series exceeds that number *N*.

In Bernoulli's Defense…

- Johann Bernoulli wrote this proof approximately 150 years before a truly rigorous theory of series was developed.
- One cannot deny the mathematical insight and cleverness of Bernoulli's argument.

Other Bernoulli Contributions

- L'Hôpital's Rule
- convergence of the sum of the reciprocals of the squares $(1+{}^{1}/_{4}+{}^{1}/_{9}+{}^{1}/_{16}+\ldots)$
- *Ars Conjectandi* (*The Art of Conjecturing*)
- Bernoulli Theorem: Law of Large Numbers
- reflection and refraction
- analytical trigonometry
- early use of polar coordinates

Conclusion

- The Bernoulli brothers made many contributions to mathematics during their lives.
- Their mathematical skills were on the cutting edge of ‘new math’ during the Seventeenth and Eighteenth Centuries.
- Where would we be mathematically without their work?

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