Applications of Linear Algebra and Statistics in Point-based Medical Image Registration

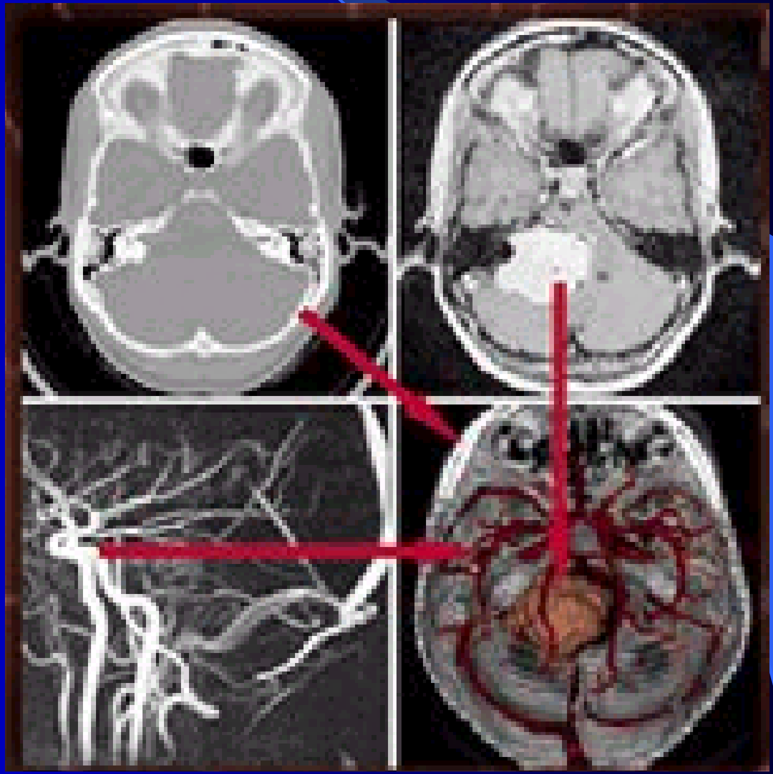
Brian A. Taylor

Department of Mathematics & Computer Science

UNION UNIVERSITY

Medical Image Registration

- Process of aligning images so corresponding features can be seen together
- Used for clinical and research purposes



Mathematical Background

- Linear transformations are used to transform one image into another.
- Statistical methods are used to determine the accuracy of the registration.

Linear Transformations

- A mapping **T** of $\mathbf{R}^n$ **into** $\mathbf{R}^m$**, written as**

$$T: \mathbf{R}^n \longrightarrow \mathbf{R}^m$$

is a rule that assigns to each vector **u** in $\mathbf{R}^n$ an unique vector **v** in $\mathbf{R}^m$

- Must be operation preserving

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

$$T(c\mathbf{u}) = cT(\mathbf{u})$$

Linear Transformations

- 2D Rotation

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- Rotate point (x,y) counterclockwise at angle θ

Linear Transformations

- Translation in the xy plane

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

Where $a, b \in \mathbf{R}$

Least Squares Method

- Given data points x_i and y_i , there are numbers a and b where

$$\sum_{i=1}^{n}[y_i - (a + bx_i)]^2$$

will give a minimum value.

Least Squares Method

$$S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$S_{yy} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

$$S_{xy} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - y)$$

Least Squares Method

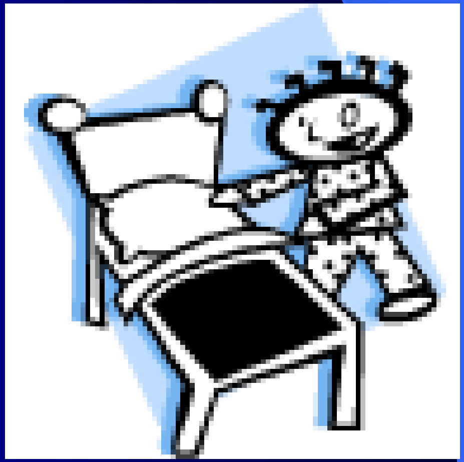
$$b = \frac{S_{xy}}{S_{xx}} \qquad a = \bar{y} - b\bar{x}$$

Error Sum of Squares

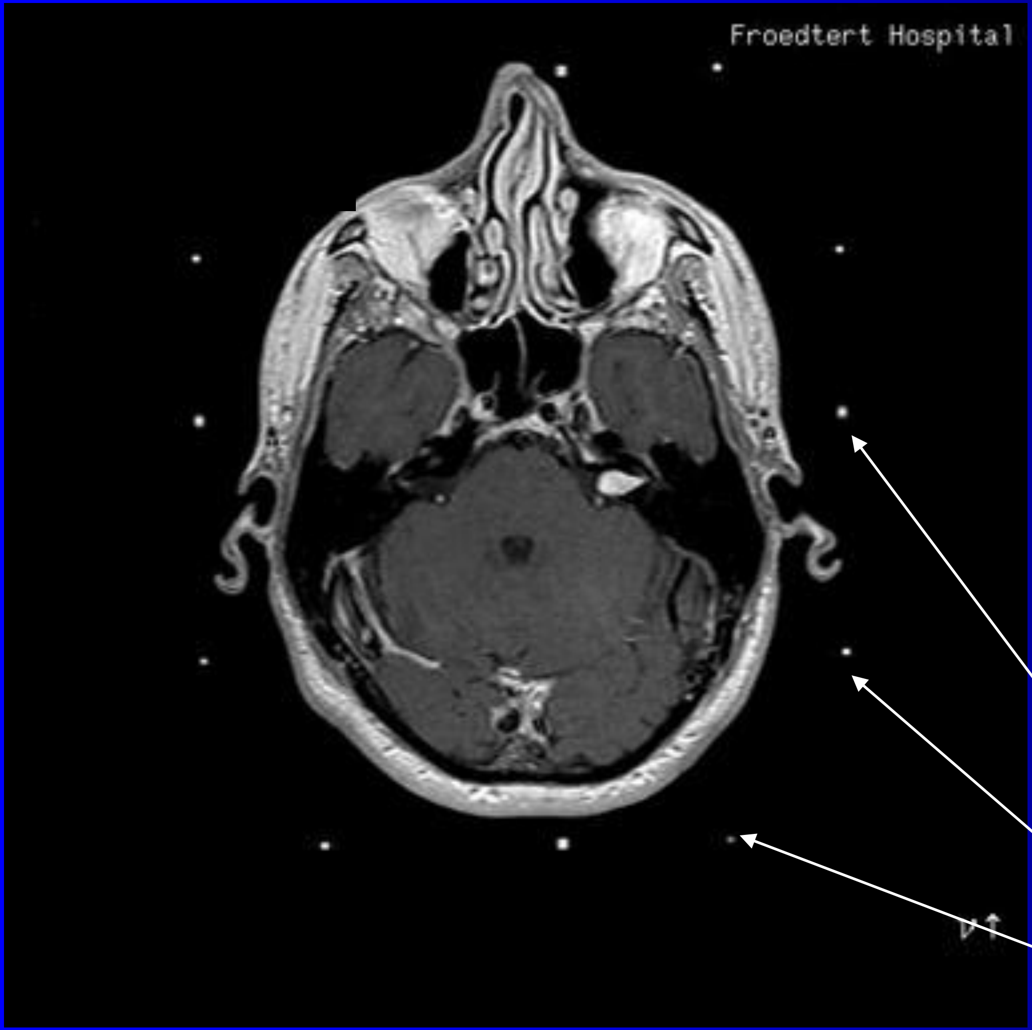
$$\sum_{i=1}^{n} (y_i - a - bx_i)^2 = S_{yy} - \frac{S_{xy}^2}{S_{xx}}$$

Orthogonal Procustes Problem

- Least Squares Method to fit one data set to another
- Solved by Schönemann in 1966
- Procustes in Greek Mythology



Point-Based Registration



Coordinates for the fiducials can be found on multiple images

One set of fiducials can be lined up with another.

Fiducials

Point-Based Registration

- Each point is represented as a vector in a column of a matrix.
- The method of least squares (in matrix form) can be used to find the rotation and translation needed to register the image.

Point-Based Registration

Given: Two 3D point sets $\{x_i\}$ and $\{y_i\}$, we wish to find the optimal rotation **R** and translation **T** where

$$y_i = \mathbf{R}x_i + \mathbf{T}$$

Registration Algorithm

First, find the centroid of each point set (x and y will be a column matrix)

$$y = \frac{1}{N}\sum_{i=1}^{N} y_i$$

$$x = \frac{1}{N}\sum_{i-1}^{N} x_i$$

Registration Algorithm

Next, sum the difference between the points and the centroid

$$q_i = x_i - x$$

$$q_i' = y_i - y$$

Minimize $\sigma^2 = \sum_{i=1}^{N} \left\| y_i - (\mathbf{R}x_i + \mathbf{T}) \right\|^2 = \frac{1}{N} \sum_{i=1}^{N} \left\| q_i' - \mathbf{R}q_i \right\|^2$

How do we minimize σ^2 ?

Answer: Find the *singular value decomposition* of

$$H = \sum_{i=1}^{N} q_i q_i'^T$$

Singular Value Decomposition (SVD)

The matrix H can be "decomposed" to

$$H = UDV^T$$

where U and V are orthonormal and D is a 3x3 diagonal matrix

$$H = UDV^T$$

$$H = UV^T$$

$$R = VU^T$$

Rotation Transformation

R will be the rotation matrix needed to register the image

$$R = VU^T$$

Note: The determinant of R must be 1. If the determinant is –1, then R contains a reflection which is not wanted.

Translation

The translation can be found simply by

$$T = y - Rx$$

Registration Errors

- Fiducial Registration Error (FRE)
- Fiducial Localization Error (FLE)
- Target Registration Error (TRE)

Fiducial Registration Error (FRE)

$$\sigma^2 = \sum_{i=1}^{N} \left\| y_i - (\mathbf{R}x_i + \mathbf{T}) \right\|^2$$

Also called the root mean squared distance

Not always an accurate measurement on the efficiency of the registration!

Fiducial Localization Error (FLE)

- Where exactly are the fiducials?
  - There can be a degree of uncertainty of where the fiducial is located.

Are the coordinates taken from the center of the fiducial or on a random location in the fiducial?

Target Registration Error (TRE)

- Determines how well areas of interest other than the fiducials are lined up.

- In 3D, the TRE can be predicted using

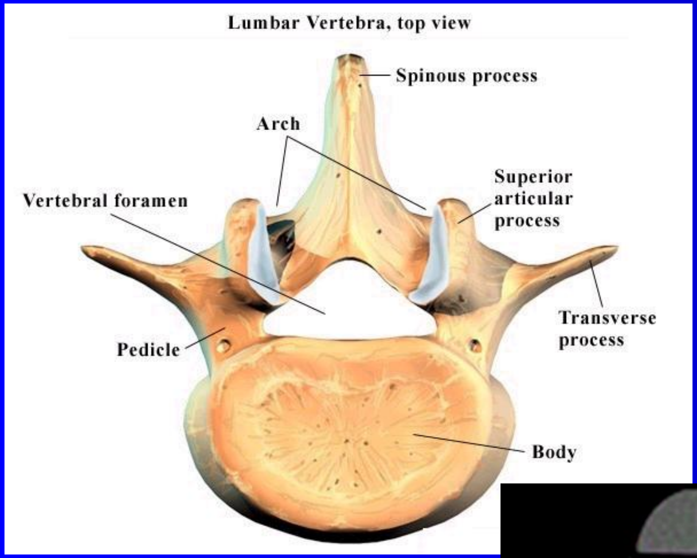
$$TRE^2(r) \approx \frac{FLE^2}{N}\left(1+\frac{1}{3}\sum_{k=1}^{3}\frac{d_k^2}{f_k^2}\right)$$

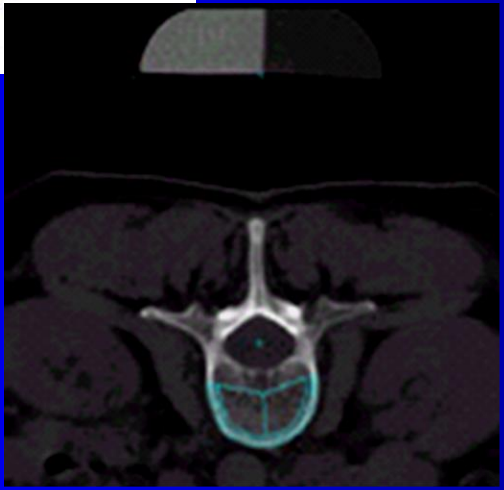
Target Registration Error

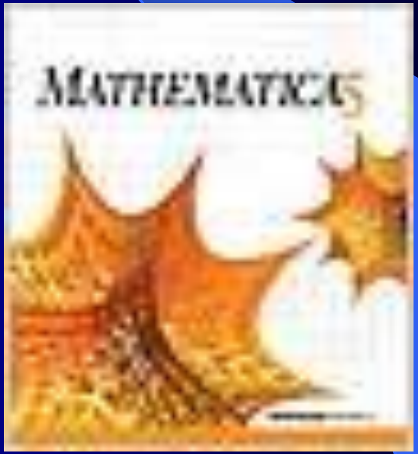
If the fiducials are on the edges (bone/skin implants) we must make sure the target is also aligned.

Target

Registration of a Spinal Vertebra Using Mathematica 5.0







Finding the Fiducials

References

Arun, K.S., T.S. Huang, and S.D. Blostein. "Least-Squares Fitting of Two 3-D Point Sets." IEEE Transactions on Pattern Analysis and Machine Intelligence PAMI-9 (1987): 698-700.

Fitzpatrick, J. Michael, Jay B. West, and Calvin R. Maurer, Jr. "Predicting Error in Rigid-body, Point-based Registration." IEEE Transactions on Medical Imaging 17 (1998): 694-702.

Hajnal, Joseph V., Derek L.G. Hill, and David J. Hawkes. Medical Image Registration. New York: CRC Press LLC, 2001.

Johnson, Richard A. Miller and Freund's Probability and Statistics for Engineers. 6th ed. Upper Saddle River, New Jersey: Prentice Hall, 2000.

Szidarovszky, Ferenc and Sándor Molnár. Introduction to Matrix Theory with Applications to Business and Economics. River Edge, New Jersey: World Scientific, 2002.

Williams, Gareth. Linear Algebra with Applications. 4th ed. Boston: Jones and Bartlett Publishers, 2001.

Acknowledgements

- Troy Riggs, Ph.D.
- Jeannette Russ, Ph.D.
- Matt Lunsford, Ph.D.
- Department of Engineering
- Department of Mathematics and Computer Science