Multiple Comparisons

The Interaction Effects of more than two factors in an analysis of variance experiment.

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One way Analysis of Variance (ANOVA)

- Testing the hypothesis that the means of k populations are all equal
- Variance between treatments of the same populations
- Using Sum of Squares, Mean Squares, F values, and α level of significance.
- Focused on the effects of one variable

Two Factor Experiments with Interaction

- Focused on the individual and joint effects of two variables.
- The different levels, or values, in this experiment will effect the calculations
- Looking at how different factors interact and effect the measured values

Collection of Data

Melt Flow Index	Filler Content	Replication 1	Replication 2	Totals
3	32	1.95	2.14	**4.09**
3	37	2.36	2.28	**4.64**
3	42	2.61	2.56	**5.17**
12	32	2.32	2.31	**4.63**
12	37	2.33	2.28	**4.61**
12	42	2.6	2.53	**5.13**
30	32	2.29	2.36	**4.65**
30	37	2.61	2.5	**5.11**
30	42	2.68	2.52	**5.2**
	Totals	**21.75**	**21.48**	**43.23**

Johnson, Richard A. Probability and Statistics for Engineers. Exercise 13.5, pg 467.

SS

SS Factors

Assumptions

- Assume a common μ and σ^2, and that the sum of every effect is equal to zero.
- Assume the Null Hypothesis that every level of each factor is equal, there is no effect.
- Assume the Alternate Hypothesis that the levels of each factor are not equal
- Assume the same significance level, α, for each factor and replication.

Critical Values

- F values are used to determine the criteria for which the test statistics will be compared to.
- This has two different degrees of freedom, one for the denominator and one for the numerator.
- Different tables for different levels of significance, α.
- For a Factorial experiment their will be F values for each of the Factors, each of the Interactions, and the Replications.
- For this example their will be four F values.

Finding Critical Values

- The numerator degrees of freedom is the degrees of freedom for the effect being tested
- The denominator degrees of freedom is the Error degrees of freedom.
- Using the Chart for a common alpha value, the F values for each effect can be obtained.

Effects and Degrees of Freedom

Effect	**Degrees of Freedom**	This Experiment
Replications	(r-1)	1
Factor A	(a-1)	2
Factor B	(b-1)	2
Interaction	(a-1)(b-1)	4
Error	(rab-1)-(a-1)(b-1)-(b-1)- (a-1)-(r-1)	8
Total	(rab-1)	17

F Values for α = 0.01

		d^1							
d^2		1	2	3	4	5	6	7	8
	1	4052	5000	5403	5625	5764	5859	5928	5982
	2	98.5	99.00	99.17	99.25	99.30	99.33	99.36	99.37
	3	34.12	30.28	29.64	28.71	28.24	27.91	27.67	27.49
	4	21.2	18	16.69	15.98	15.52	15.21	14.98	14.8
	5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29
	6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.1
	7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84
	8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03

Calculating F Values

- For the Main Factor A

  $d^1=2$, $d^2=8$, at $\alpha=.01$, **F=8.65**

- For the Main Factor B

  $d^1=2$, $d^2=8$, at $\alpha=.01$, **F=8.65**

- For the Replication

  $d^1=1$, $d^2=8$, at $\alpha=.01$, **F=11.26**

- For the Interaction

  $d^1=4$, $d^2=8$, at $\alpha=.01$, **F=7.01**

Calculations

Name	Formula	Notation

Correction Term	$C = (T..)^2/(r*a*b)$	r = # replications,
		a= # levels of factor a
		b= # levels of factor b
		T = total values for treatment

Sum of Squares Total	$SST = \sum\sum\sum y^2 - C$	i=1 to n, j=1 to n, and ℓ=1 to r, for i, j(k), ℓ
Sum of Squares Treatment	$SS(Tr) = 1/(r)\sum T^2 - C$	k to n, for k
Sum of Squares Replications	$SSR = 1/(a*b)\sum T^2 - C$	i to n, for i
Sum of Squares Error	SSE = SST – SS(Tr) – SSR	
Sum of Squares Factor A	$SSA = 1/(b*r)\sum T^2 - C$	i to a, for i level of factor
Sum of Squares Factor B	$SSB = 1/(a*r)\sum T^2 - C$	j to b, for j
Sum of Squares Interaction of A and B	SS(AB) = SS(Tr) – SSA - SSB	
Mean Square	SS (X) / Degrees of freedom	X = effect being measured
Test Statistics	Mean square / Error Mean Square	

Applying the Formulas

- $C=(1.95 + 2.14 + 2.36 + \ldots + 2.52)^2/(3*3*2)$
  $C=(43.23)^2/18$

  $C = 103.8241$

- $SST = (1.95^2) + (2.14^2) + \ldots + (2.52^2) - 103.8241$

  $SST = .60705$

- $SS(Tr) = (1/2)*(4.09^2 + 4.64^2 + \ldots + 5.2^2) - 103.8241$

  $SS(Tr) = .5595$

- $SSR=(1/(3*3))(21.75^2 + 21.48^2) - 103.8241$

  $SSR= .00405$

- $SSE= .60705 - .5595 - .00405$

  $SSE= .0435$

Data Table

Finding the SSA, SSB, and SS(AB)

- Create the two way table for the totals of factor A and B.

	Factor B (Filler Content)				Totals
		32	37	42	
Factor A (Melt Flow Index)	3	4.09	4.64	5.17	**13.9**
	12	4.63	4.61	5.13	**14.37**
	30	4.65	5.11	5.2	**14.96**
Totals		**13.37**	**14.36**	**15.5**	**43.23**

- Click button to return to Data Table

Data Table

Calculating SSA, SSB, and SS(AB)

- $SSA = (1/(3*2))*((13.9)^2 + (14.37)^2 + (14.96)^2) - 103.8241$

SSA = .0939833

- $SSB = (1/(3*2))*((13.37)^2 + (14.36)^2 + (15.5)^2) - 103.8241$

SSB = .37865

- SS(AB) = .5595 - .0939833 - .37865

SS(AB) = .0868667

Find the Mean Squares

- Taking each of the Sum of Squares and dividing the by their degrees of freedom to obtain the Mean Square.
- The test statistic is found by taking the Mean Square and dividing it by the Error Mean Square.
- This is done for each of the Main Factor, the Interaction, and the Replication effects.

Calculating the Mean Squares

- MS(R) = .00405/1

  MS(R) = .00405
- MS(A) = .0939833/2

  MS(A) = .04699165
- MS(B) = .37865/2

  MS(B) = .189325
- MS(AB) = .0868667/4

  MS(AB) = .021716675
- MS(E) = .0435/8

  MS(E) = .0054375

Calculating F test statistics

- Replication F = .00405/.0054375

  RF = .744828
- Factor A F = .04699165/.0054375

  AF = 8.64214
- Factor B F = .189325/.0054375

  BF = 34.81839
- Interaction F = .021716675/.0054375

  ABF = 3.99387

Comparing F Values

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	Critical Values	Test Statistics
Replications	1	.00405	.00405	11.26	.744828
Factor A	2	.0939833	.04699165	8.65	8.64214
Factor B	2	.37865	.189325	8.65	34.81839
Interaction	4	.0868667	.021716675	7.01	3.99387
Error	8	.0435	.0054375		
Total	17	.60705			

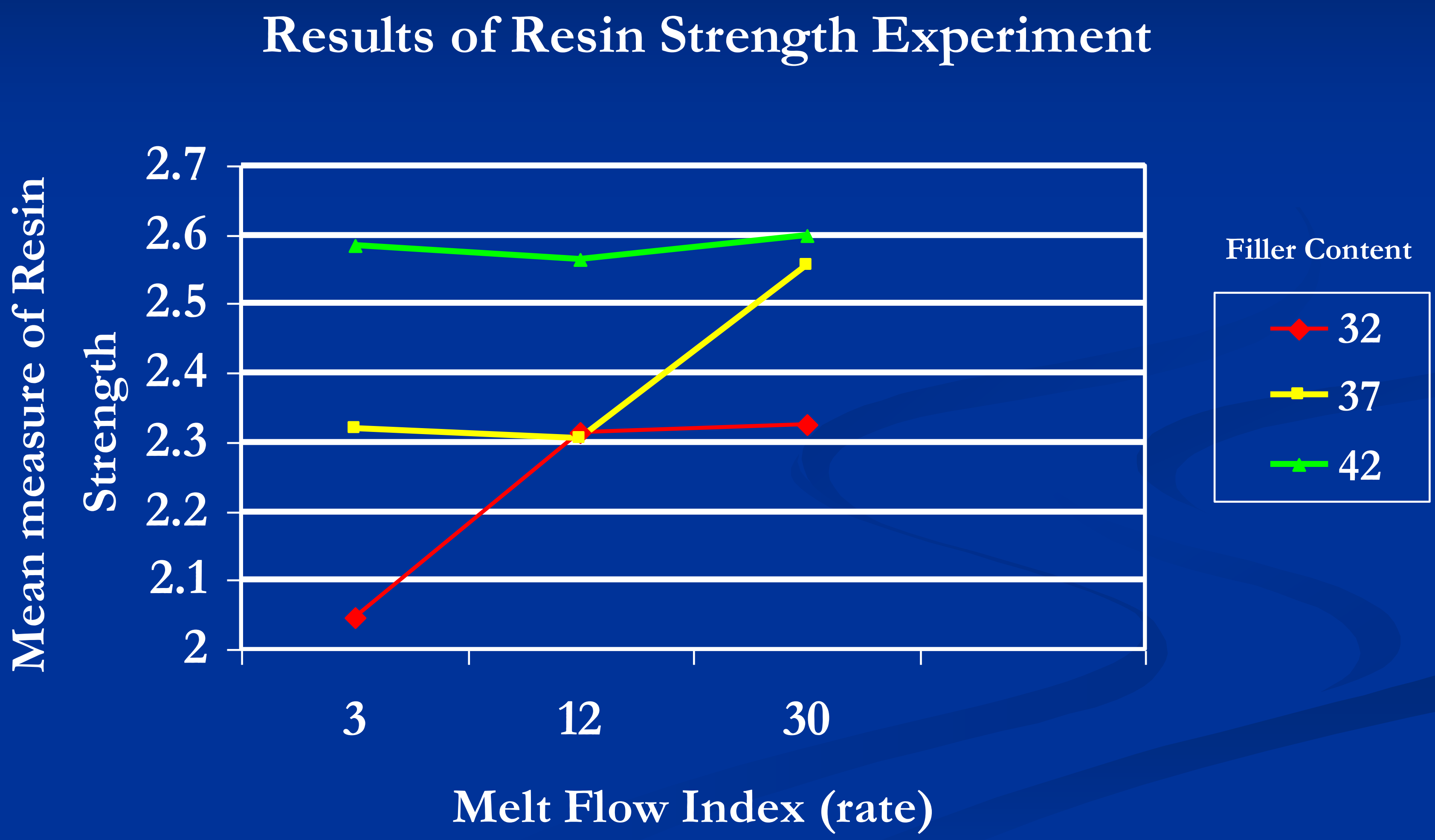
Summarizing Test

- Factor A's F value is significant at the $\alpha=.01$ level because AF>F.
- Likewise, B's F Value is significant at the $\alpha=.01$ level because BF>F.
- However, at the $\alpha=.01$ level the RF and ABF are not significant.
- Since this test was done with such a high confidence, we will look at $\alpha=.05$.

F for α=.05

- From a similar table as before it can be found that the critical value for the Replications and Interactions at α=.05 are
  - Replication, **F=5.32**
  - Interaction, **F=3.84**
- Thus, the Interaction's F statistic is significant at the α=.05 level because ABF>F
- The Replication's F statistic is however, not significant at the α=.05 level

Results


Results of Resin Strength Experiment
Mean measure of Resin Strength
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2
3
12
30
Melt Flow Index (rate)
Filler Content
32
37
42

Results

- Another way to show the results is with a Table of Cell Means

	Factor B (Filler Content)				Totals
		32	37	42	
Factor A (Melt Flow Index)	3	2.045	2.32	2.585	6.95
	12	2.315	2.305	2.565	7.185
	30	2.325	2.555	2.6	7.48
Totals		6.685	7.18	7.75	21.615

Two Way Analysis with Excel

- Excel can also work these problems quickly and accurately.
- The button below will take us to an example on Excel.

Excel Workbook

Multi-Factor Experiments

- Once Two-way analysis is understood, it is simple to see how these experiments and calculations can be altered for more than 2 factors.
- A general example of a 3 factor experiment, can illustrate the necessary changes for any amount of factors

3 Factor Experiment

- 3 Factors
  - A has 4 levels
  - B has 3 levels
  - C has 2 levels
- We will have 24 (2*3*4) rows of data
- There are 2 replications for this data set
- Measuring the Electric Conductivity

Factor	Level 1	Level 2	Level 3	Level 4
A. Acid Concentration (%)	0	6	12	18
B. Salt Concentration (%)	0	10	20	
C. Bath Temperature (°F)	80	100		

Data

- As more factors are added to the experiment, the table for showing the data will also grow.
- The data for this example is too large to show on this slide
- The following button will link us to the Excel sheet that holds the data

3 Factor Example

Degrees of Freedom

- For each factor it is the same (# levels – 1)
- For interactions it is still (# levels A – 1)*(# levels B -1), but their will be an interaction for AB, BC, and AB.
- There will also be an interaction ABC with (# levels A – 1)*(# levels B – 1)*(# levels C – 1)
- In this 3 factor example
  - R = 1
  - A = 3
  - B = 2
  - C = 1
  - AB = 6
  - AC = 3
  - BC = 2
  - ABC = 6
  - Total = 47
  - Error = 23

Sum of Square for Interaction

- Two way tables of interaction for the totals of each factors interactions

		Factor A				Total
		1	2	3	4	
Factor C	1	5.61	6.12	6.6	6.71	**25.04**
	2	5.63	6.12	6.41	6.97	**25.13**
Total		**11.24**	**12.24**	**13.01**	**13.68**	**50.17**

		Factor B			Total
		1	2	3	
Factor A	1	4.06	3.61	3.57	**11.24**
	2	4.42	3.97	3.85	**12.24**
	3	4.73	4.16	4.12	**13.01**
	4	5	4.63	4.05	**13.68**
Total		**18.21**	**16.37**	**15.59**	**50.17**

		Factor C		Total
		1	2	
Factor B	1	8.99	9.22	**18.21**
	2	8.25	8.12	**16.37**
	3	7.8	7.79	**15.59**
Total		**25.04**	**25.13**	**50.17**

Back

Sum of Squares for Factors

- Equal to the sum of the totals squared of the factor being looked at divided by the product of the number of levels of the factors not being observed and the number of replications minus the correction factor.
- Including a total treatment sum of squares the multifactor experiment will need treatment sum of squares for each of the two way tables of interactions.
- The Sum of squares for interactions are found by taking the treatment sum of squares for that interaction and subtracting each of the factor sum of squares included in that interaction.

Sum of Squares

3 Factor Sum of Squares

- The three way interaction is found by taking the treatment sum of squares minus each of the sum of squares for interactions and each of the sum of squares of each factor.

Sum of Squares	
C	52.43810208
SS(Tr)	0.571147917
SST	0.662397917
SSR	0.016502083
SSE	0.074747917
SS(TrAB)	0.530072917
SS(TrAC)	0.283714583
SS(TrBC)	0.230585417
SSA	0.275039583
SSB	0.226216667
SSC	0.00016875
SS(AB)	0.028816667
SS(BC)	0.0042
SS(AC)	0.00850625
SS(ABC)	0.0282

Mean Squares

- Each mean square is still found by the Sum of Squares divided by the degree of freedom.
- There are now three 2 way interaction effects and one 3 way interaction effect.

Mean Square	
MSR	0.016502
MSE	0.00325
MSA	0.09168
MSB	0.113108
MSC	0.000169
MS(AB)	0.004803
MS(BC)	0.0021
MS(AC)	0.002835
MS(ABC)	0.0047

F values

- As before the F test statistics are found by taking the Mean Squares and dividing by the Error Mean Square.
- Comparing the test statistics to the critical values at an $\alpha=.05$ level of significance it can be seen which factors and interactions are significant.

Source of Variation	Test Statistics	Critical Values
Replications	5.077705622	4.28
Factor A	28.20997427	3.03
Factor B	34.8035341	3.42
Factor C	0.051924524	4.28
Interaction AB	1.477818966	2.53
Interaction BC	0.646171855	3.42
Interaction AC	0.872460213	3.03
Interaction ABC	1.446194153	2.53

Results

- Replication Effect, Factor A, and Factor B are significant, however, Factor C and all the Interaction Effects are not significant.
- In multi factor experiments when factors are shown to not be significant the experiment can be made more simple by combining, or lumping, together factors.

Statistical Programs

- SAS – Purchasable, used by many company and universities for statistical analysis
- SPSS - Purchasable, more user friendly than SAS
- R - a free downloadable program

Applications of Factorial Experiments

- Efficiency in techniques and/or development
- Surveys with demographics
  - Must have a common measurement
  - Typically a test survey
  - Example from Jump$tart©

Sample
Survey

References

- Johnson, Richard A. Probability and Statistics for Engineers. 7th ed. Upper Saddle River, NJ: Pearson Prentice Hall, 2005. 397-470.
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- What Statistical Analysis Should I Use? Statistical Analyses Using Stata. UCLA Academic Technology Services. 17 Nov. 2006 <Http://www.ats.ucla.edu/stat/stata/whatstat/whatstat.htm>.
- Casserly, David, comp. 2006 Survey of Financial Literacy Among High School Students (with Stats and Answers). 5 Apr. 2006. The Jump$tart Coalition for Personal Financial Literacy. 3 Dec. 2006 <http://www.jumpstart.org/download.cfm>.