THE PROOF IS IN THE PICTURE:

Effectiveness of the Usage of Visual Illustrations to Decipher Proofs

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What is a proof?

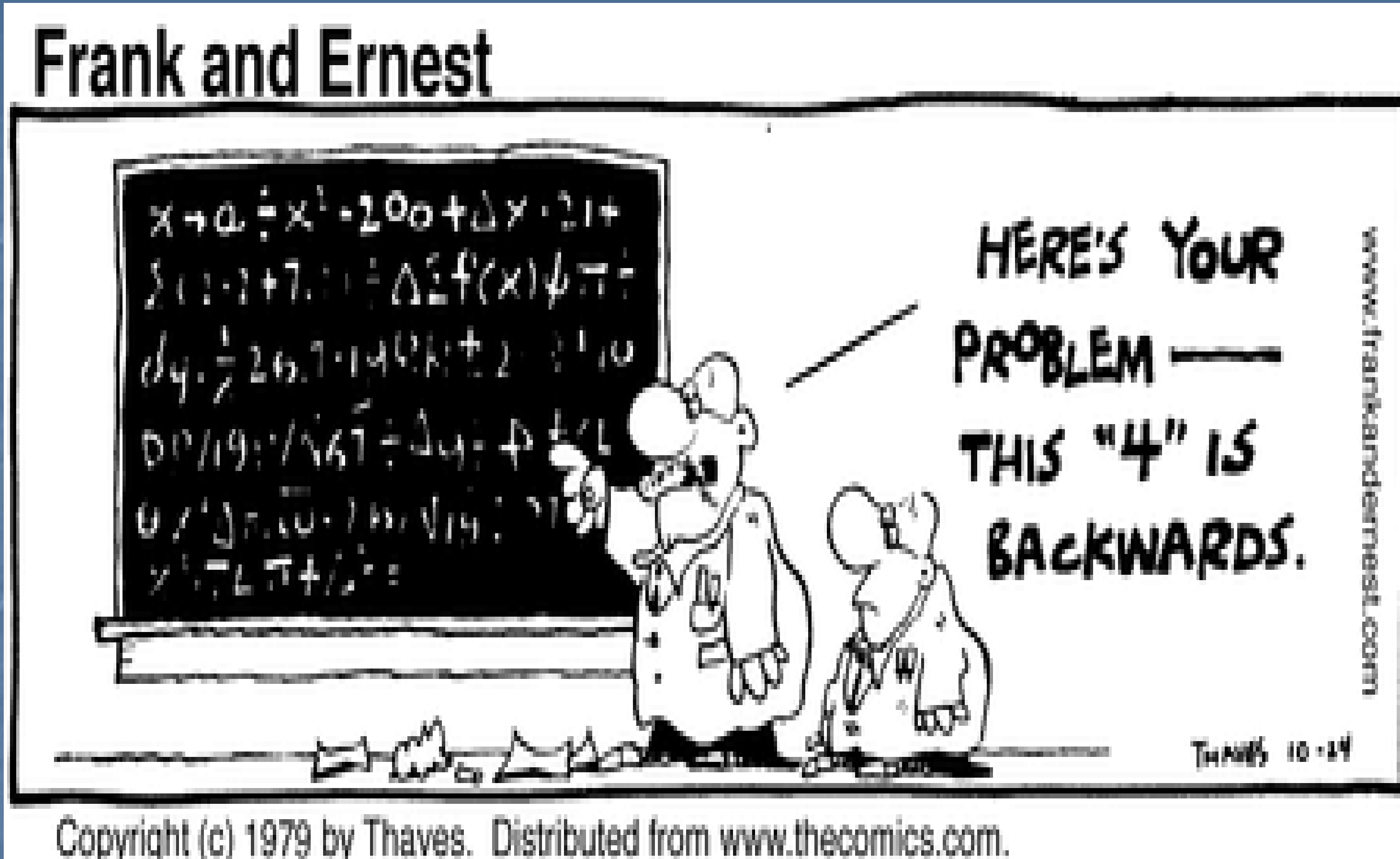
- Eugenia Cheng in *How to Write Proofs: A Quick Guide* defines a mathematical proof as "a series of statements, each of which follows logically from what has gone before."
- In other words, a proof is a demonstration of the validity of a particular mathematical concept.

Steps for Writing a Proof

- 1. Copy the statement of the theorem to be proved
- 2. Clearly mark the beginning of the proof with the word <u>Proof</u>
- 3. Write the steps in complete sentences
- 4. Clearly identify and define the variables used in the proof
- 5. Give a reason or principle that supports the assertion being made
- 6. The last step involves including terms such as therefore, thus or the, to the argument

Why Proofs are essential?

- Without proofs, mathematical concepts would not be justified. They would merely be statements that have not been linked to a solid argument.

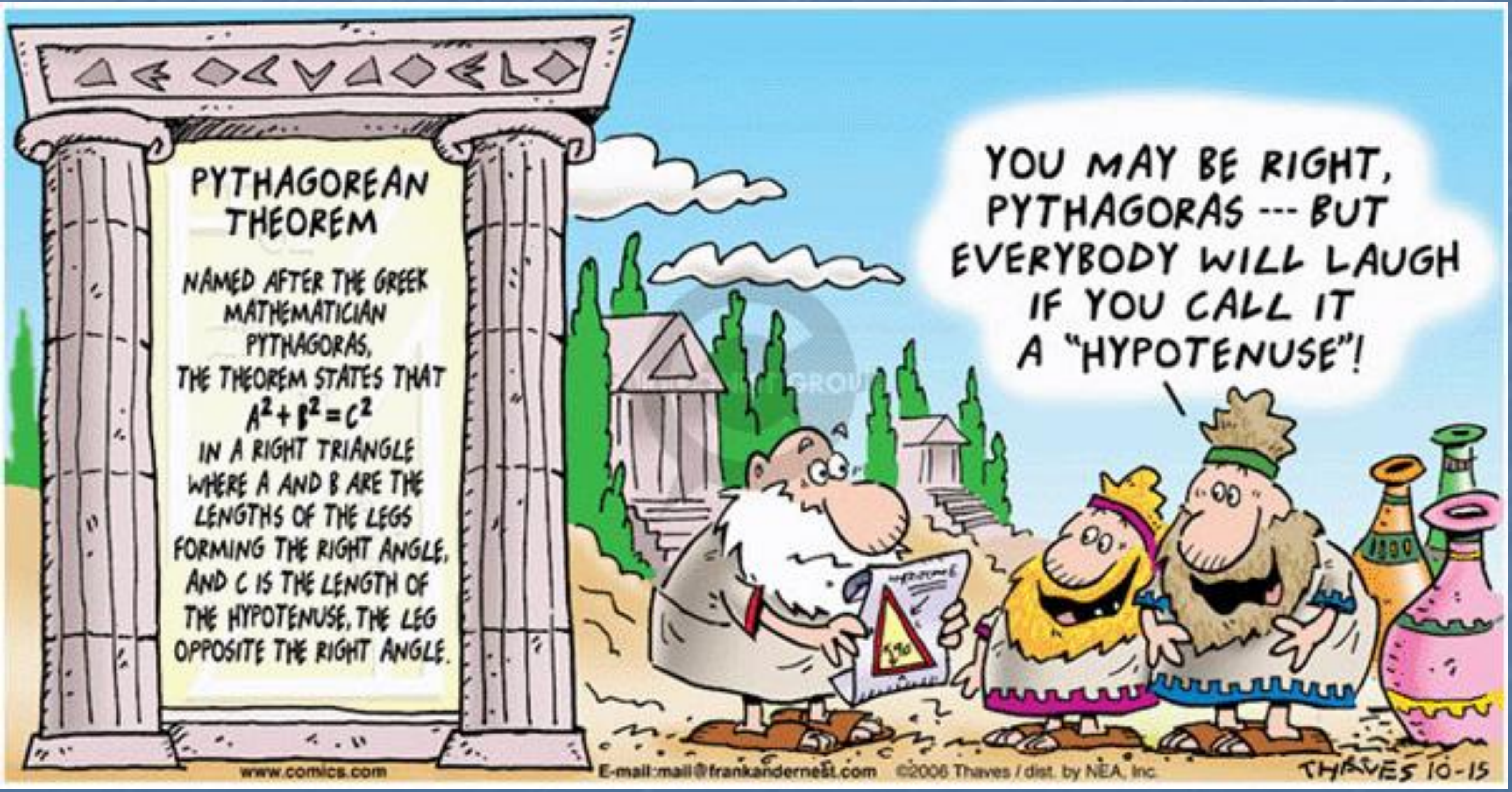


The Proof is in the Picture

How can pictures be of assistance when deciphering a proof?

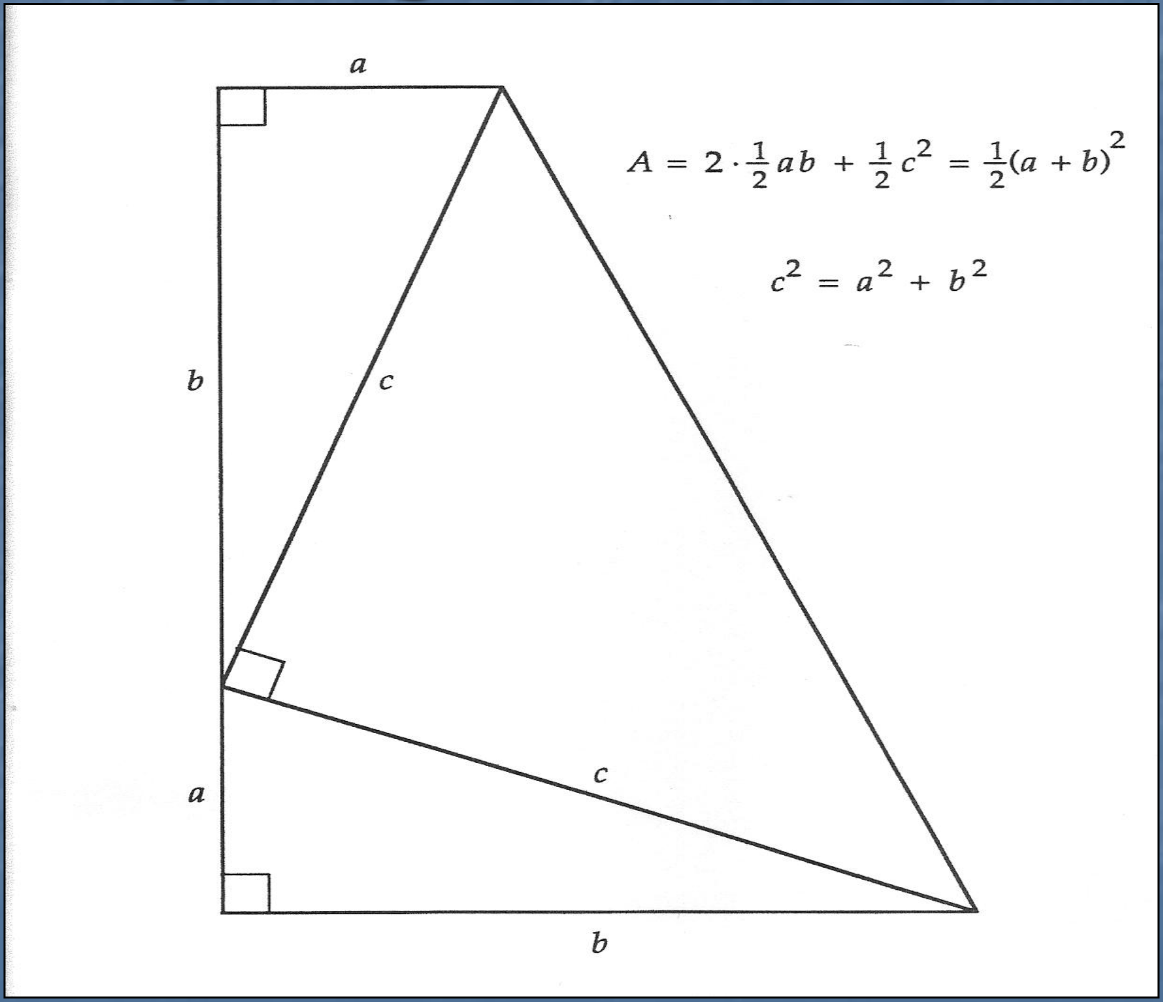
- Pictures appeal to the senses.
- They appeal to visual learners that understand a concept better when it is accompanied with a visual.
- They are creative and unique for that proof.
- They also make abstract concepts tangible and easier to analyze.
- They assist students enrich their ability to analyze visual representations.

The Pythagorean Theorem



The Proof is in the Picture

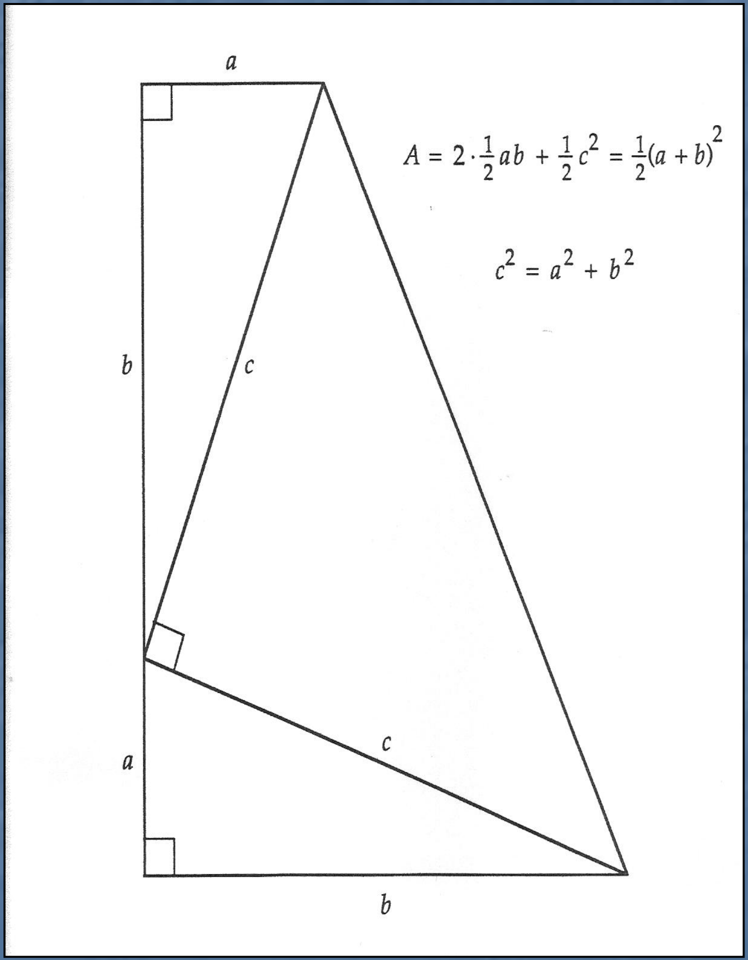
The Pythagorean Theorem



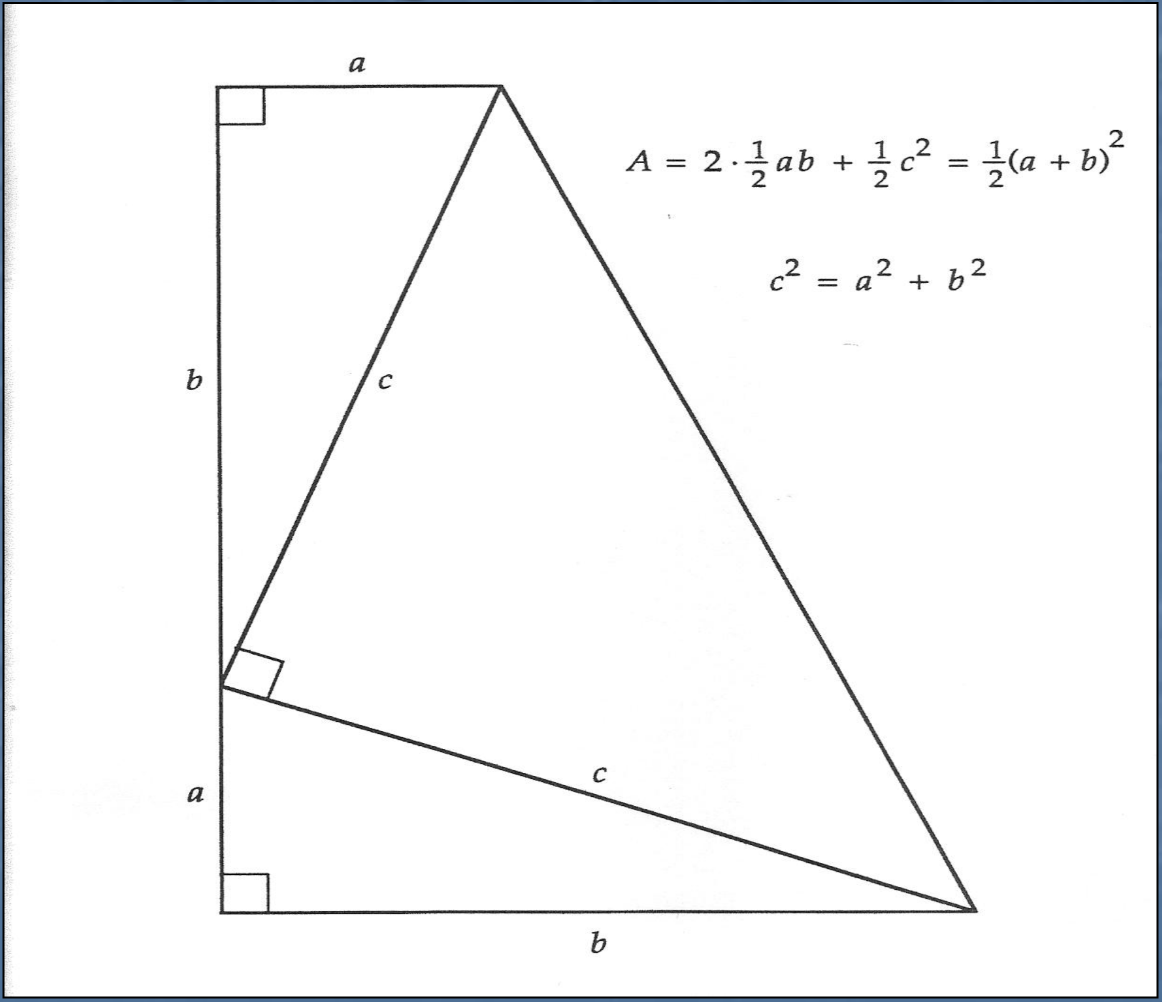
The Proof is in the Picture

Deciphering the Illustration

- Three right triangles
- Area of a triangle = ½ b h
- Area of a Trapezoid = ½ (b + a) h

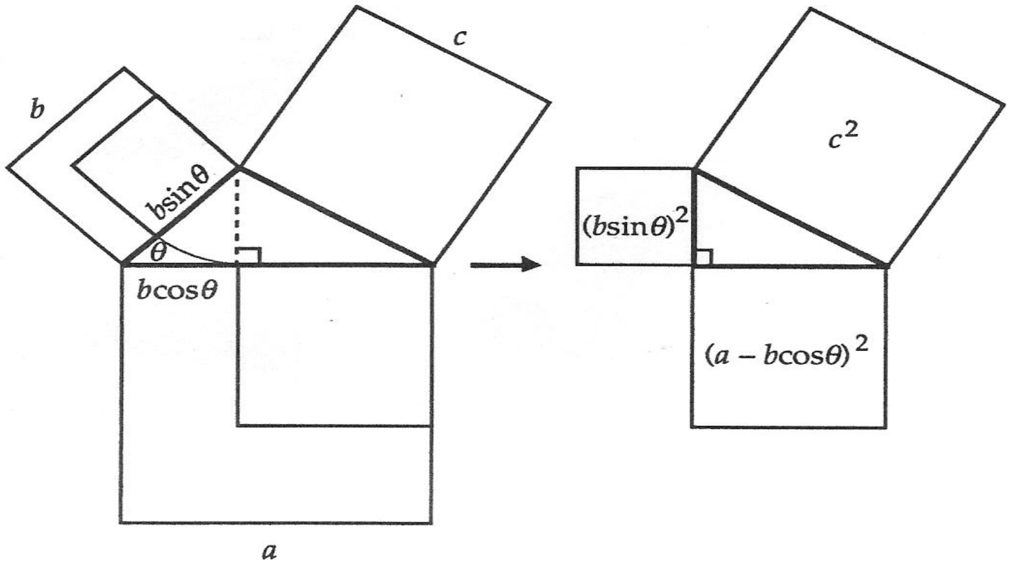


The Pythagorean Theorem



The Proof is in the Picture

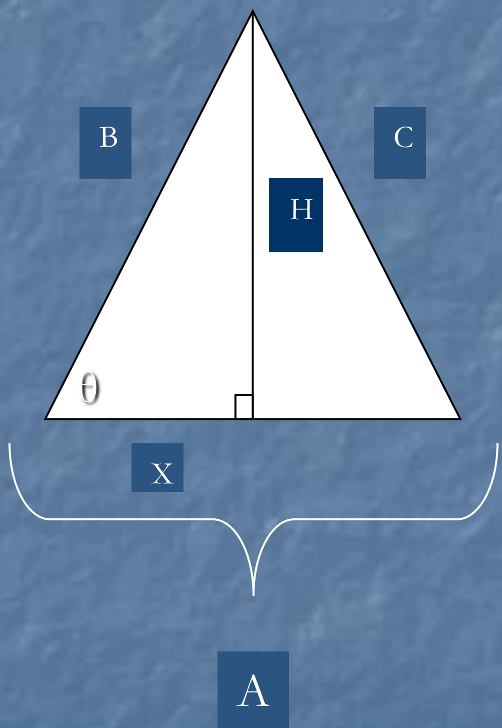
The Law of Cosines I



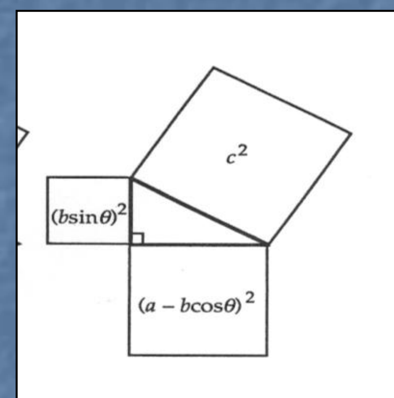
$$c^2 = (b \sin \theta)^2 + (a - b \cos \theta)^2$$
$$= a^2 + b^2 - 2ab \cos \theta$$

The Proof is in the Picture

Deciphering the Illustration



- H = B Sin θ
- X = B Cos θ



$$c^2 = (b \sin \theta)^2 + (a - b \cos \theta)^2$$
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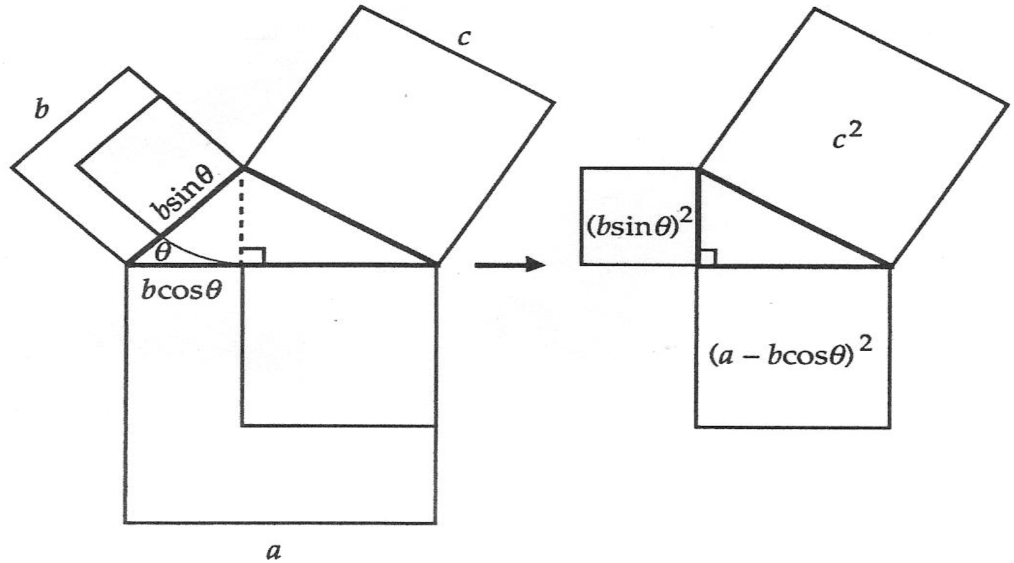
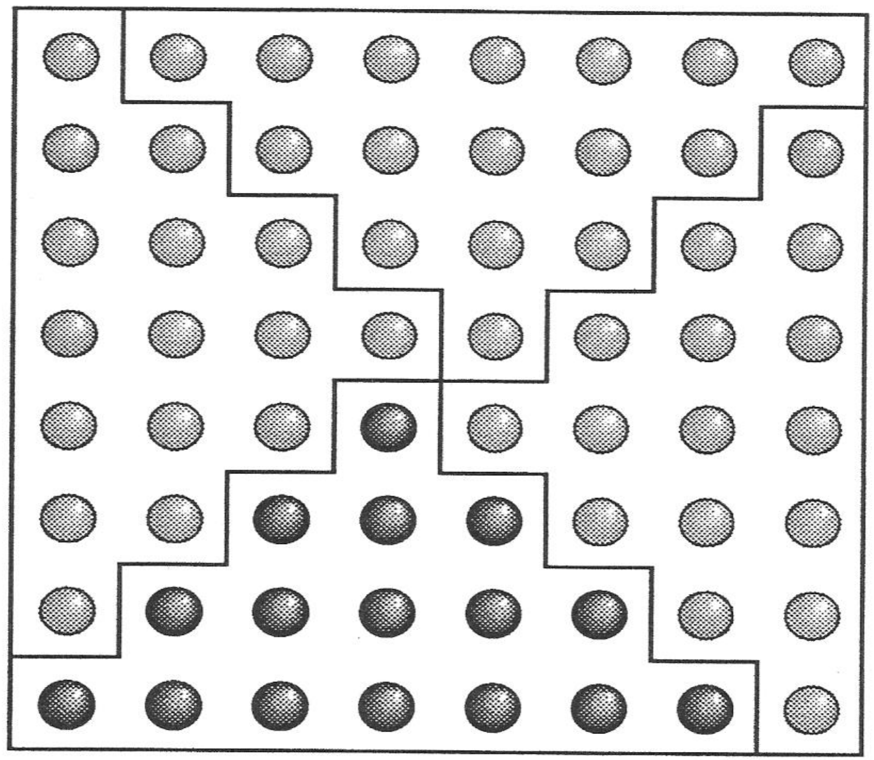
The Law of Cosines I



$$\begin{aligned} c^2 &= (b \sin \theta)^2 + (a - b \cos \theta)^2 \\ &= a^2 + b^2 - 2ab \cos \theta \end{aligned}$$

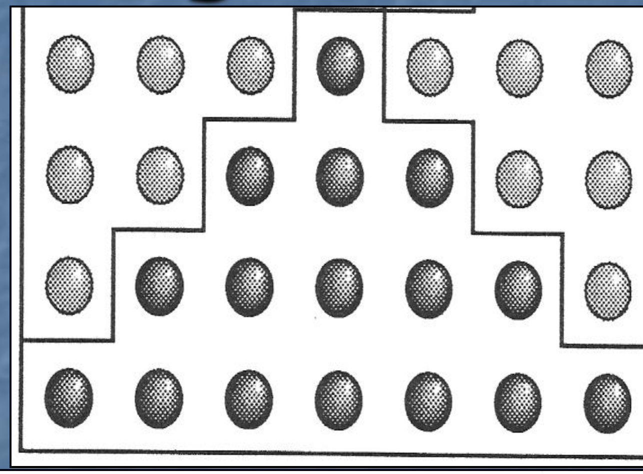
The Proof is in the Picture

Sum of Odd Integers II



$$1 + 3 + \cdots + (2n-1) = \frac{1}{4}(2n)^2 = n^2$$

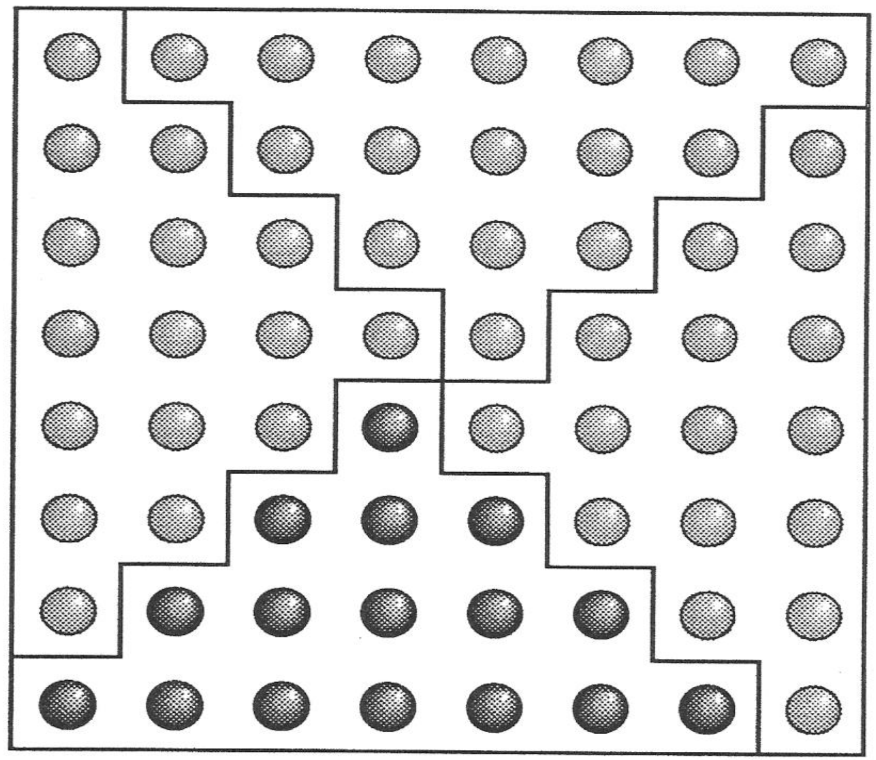
Deciphering the Illustration



$$1 + 3 + \cdots + (2n - 1) = \frac{1}{4}(2n)^2 = n^2$$

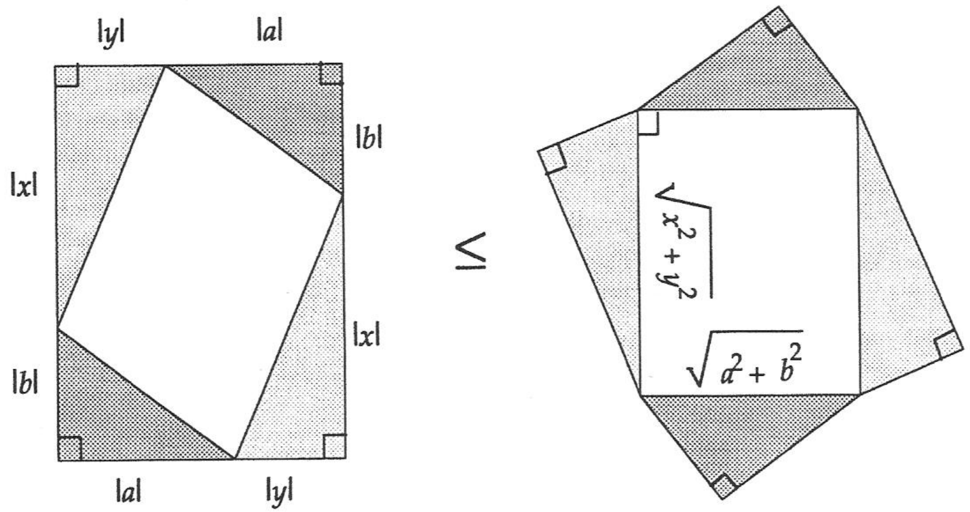
- Use of shading
- N represents the length and also the width of each side of the rectangle.
- Therefore, the area is 2N.
- ¼ $(2N)^2$ is the result of taking ¼ of the entire rectangle of N width and N length.

Sum of Odd Integers II



$$1 + 3 + \cdots + (2n-1) = \frac{1}{4}(2n)^2 = n^2$$

Cauchy – Schwarz Inequality

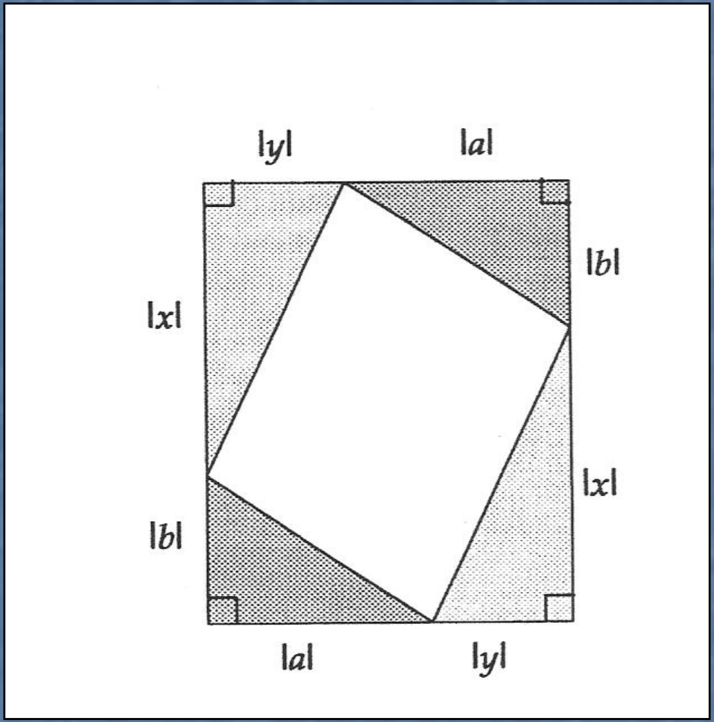


$$(|a|+|y|)(|b|+|x|) \leq 2\left(\frac{1}{2}|a||b| + \frac{1}{2}|x||y|\right) + \sqrt{a^2+b^2}\sqrt{x^2+y^2}$$

$$\therefore\ |ax+by| \leq |a||x| + |b||y| \leq \sqrt{a^2+b^2}\sqrt{x^2+y^2}$$

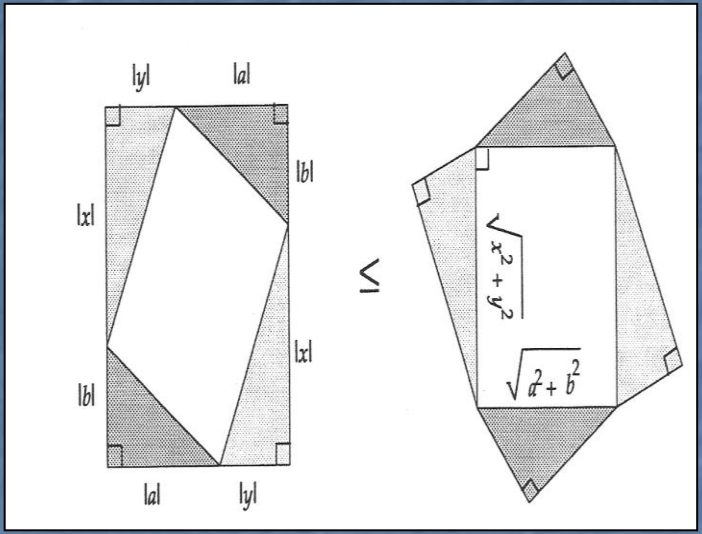
The Proof is in the Picture

Deciphering the Illustration



- Parallelogram surrounded by triangles on each side
- length of each side is (|a+y|) and (|x +b|)
- Total area –

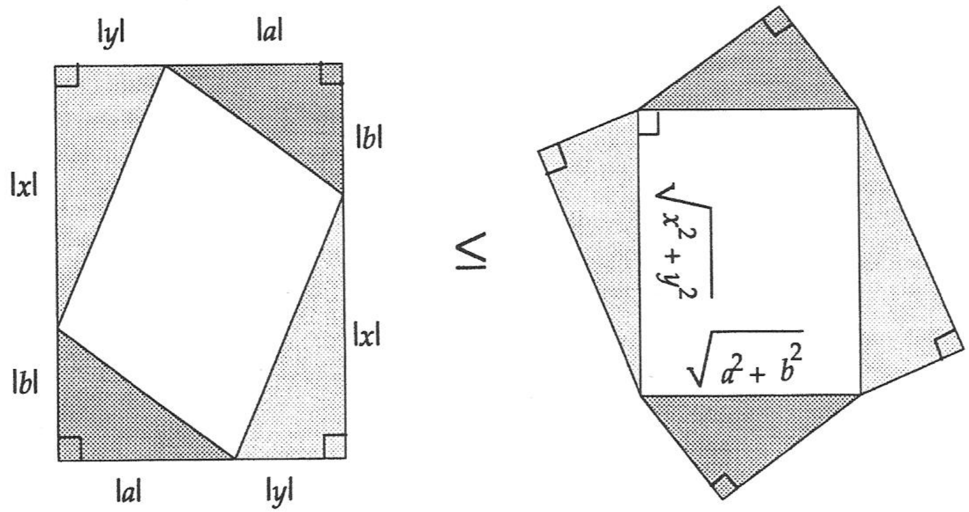
  (|a+y|) (|x +b|)



- By assuming the Triangle inequality -

  $|ax + by| \le |a||x| + |b||y|$.

Cauchy – Schwarz Inequality

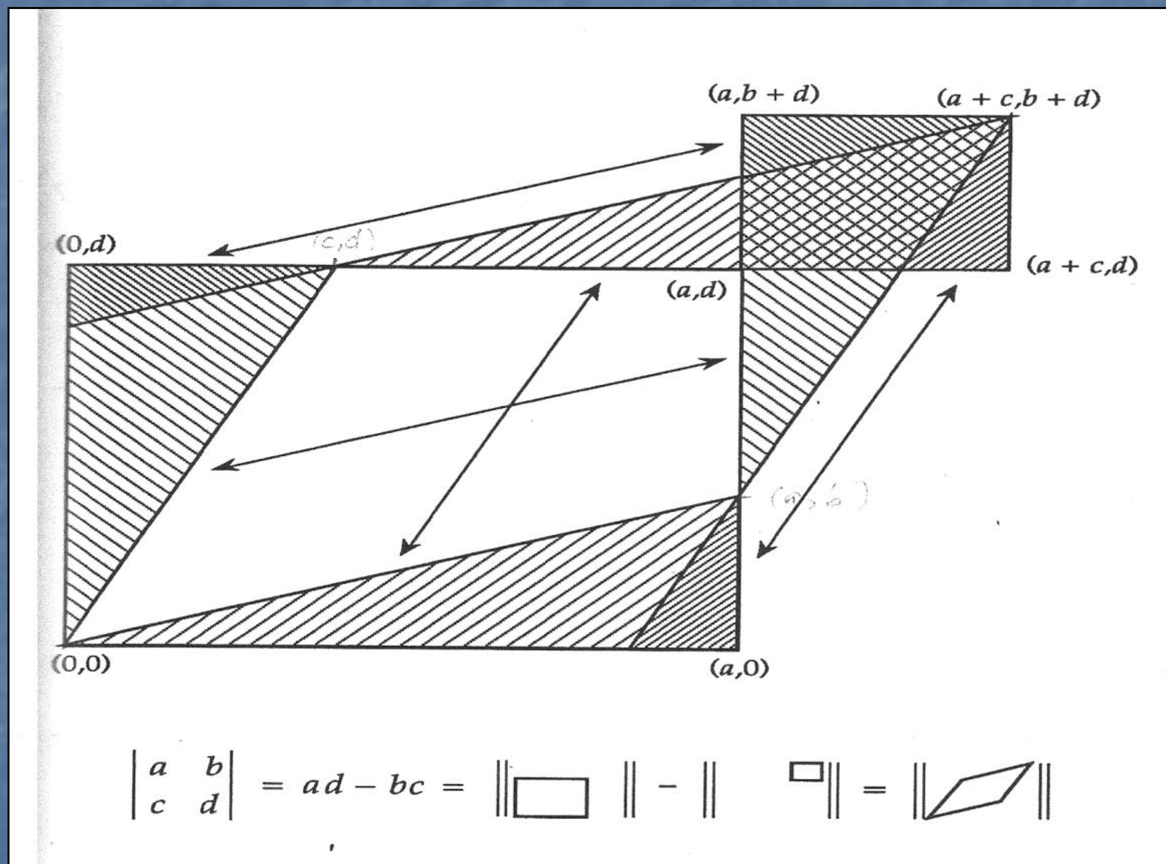


$$(|a| + |y|)(|b| + |x|) \leq 2\left(\frac{1}{2}|a||b| + \frac{1}{2}|x||y|\right) + \sqrt{a^2 + b^2}\sqrt{x^2 + y^2}$$

$$\therefore\ |ax + by| \leq |a||x| + |b||y| \leq \sqrt{a^2 + b^2}\sqrt{x^2 + y^2}$$

The Proof is in the Picture

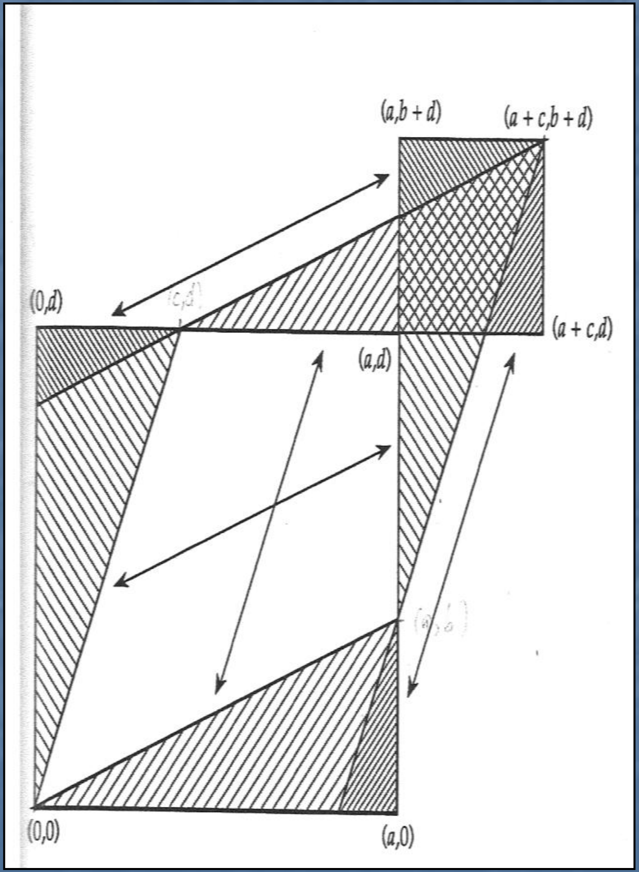
A 2 x 2 Determinant is the Area of a Parallelogram



The Proof is in the Picture

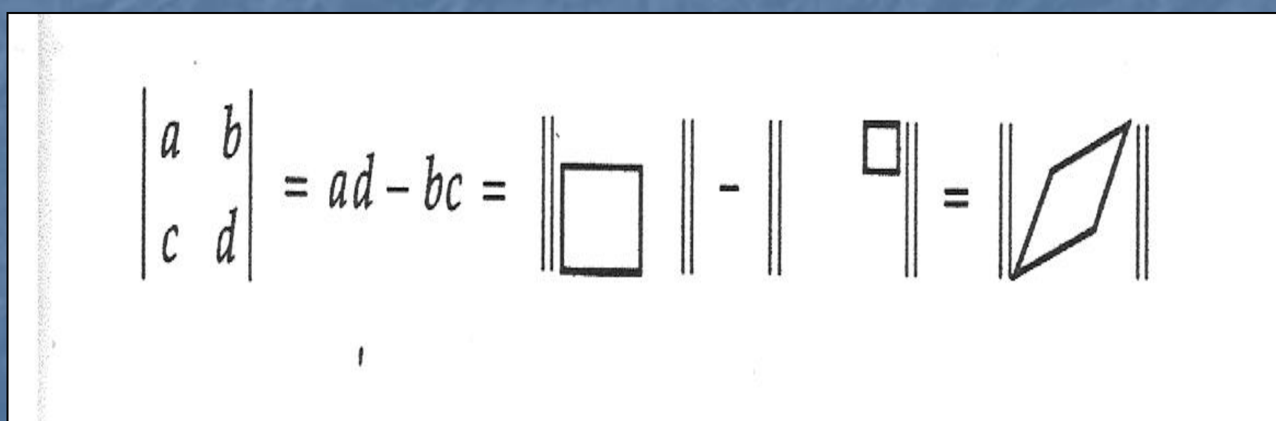
Deciphering the Illustration

- Arrows are used to show the new placement of the triangles.

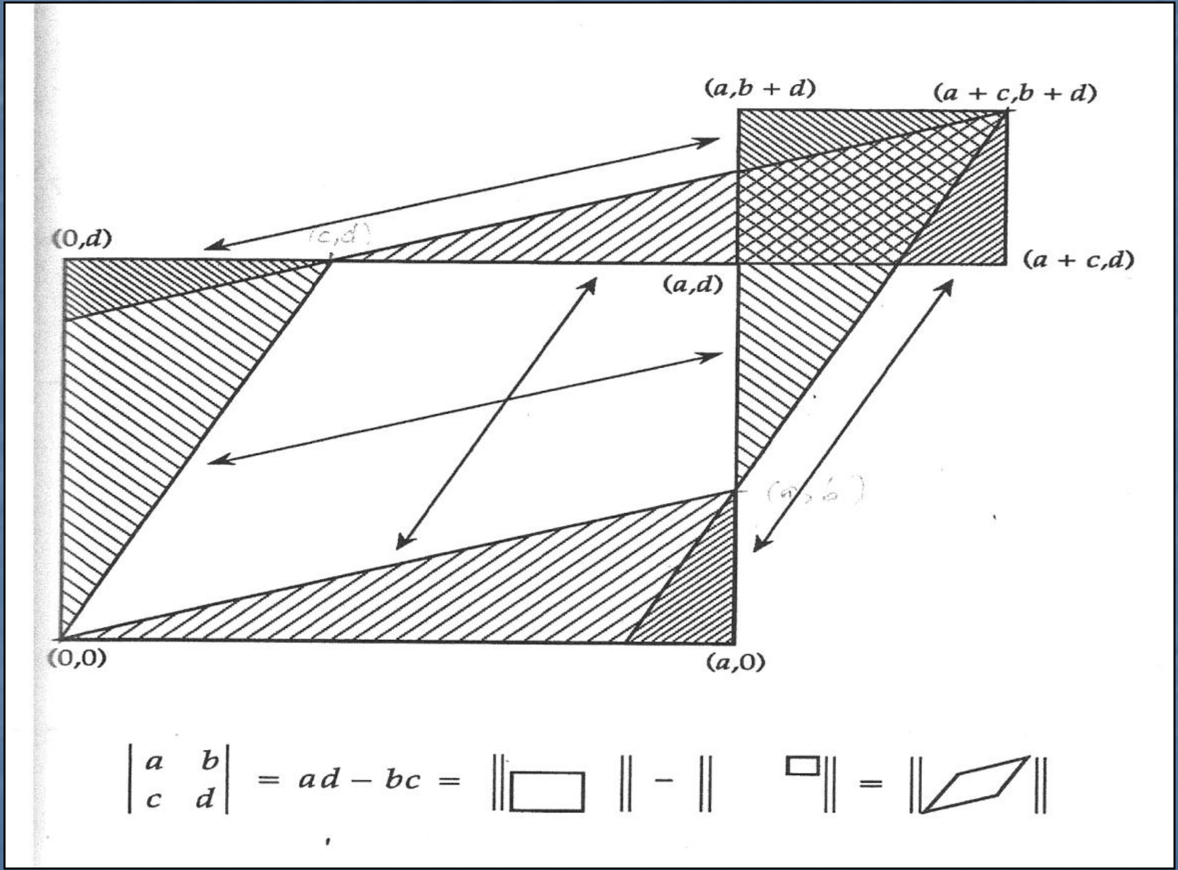


Deciphering the Illustration

- One of the strengths of the picture is that it clearly gives an equation that shows that if the smaller rectangle is subtracted from the larger a parallelogram will be the result.

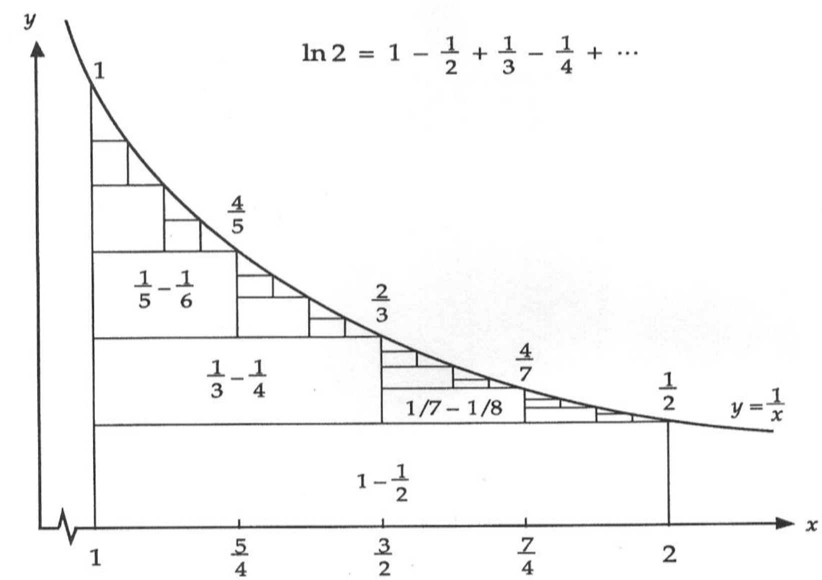


A 2 x 2 Determinant is the Area of a Parallelogram



The Proof is in the Picture

Alternating Harmonic Series



$$\frac{1}{2}\left(\frac{2}{3}-\frac{2}{4}\right)=\frac{1}{3}-\frac{1}{4};$$

$$\frac{1}{4}\left(\frac{4}{5}-\frac{4}{6}\right)=\frac{1}{5}-\frac{1}{6},\qquad \frac{1}{4}\left(\frac{4}{7}-\frac{4}{8}\right)=\frac{1}{7}-\frac{1}{8};$$

$$\frac{1}{2^n}\left(\frac{2^n}{2^n+2k-1}-\frac{2^n}{2^n+2k}\right)=\frac{1}{2^n+2k-1}-\frac{1}{2^n+2k},\quad \begin{array}{l} k=1,2,\cdots,2^{n-1}; \\ n=1,2,\cdots. \end{array}$$

$$\ln 2=\int_1^2\frac{dx}{x}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$$

—Mark Finkelstein

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Deciphering the Illustration

$$\frac{1}{2}\left(\frac{2}{3}-\frac{2}{4}\right)=\frac{1}{3}-\frac{1}{4};$$

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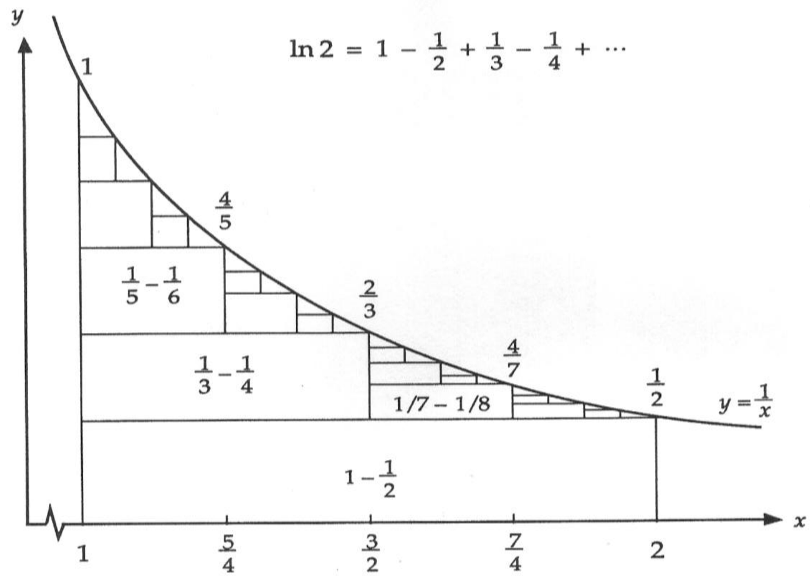
$$\frac{1}{2^n}\left(\frac{2^n}{2^n+2k-1}-\frac{2^n}{2^n+2k}\right)=\frac{1}{2^n+2k-1}-\frac{1}{2^n+2k},\quad \begin{array}{l} k=1,2,\cdots,2^{n-1};\\ n=1,2,\cdots.\end{array}$$

$$\ln 2=\int_1^2\frac{dx}{x}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$$

—Mark Finkelstein

- Obtain the sum of the area under the curve by adding the rectangles
- Pattern that exists – (base) (height)
- Numerators – powers of 2
- Denominators – 2^n – 2K – 1
- Area under the curve = the alternating harmonic series

Alternating Harmonic Series



$$\frac{1}{2}\left(\frac{2}{3} - \frac{2}{4}\right) = \frac{1}{3} - \frac{1}{4};$$

$$\frac{1}{4}\left(\frac{4}{5} - \frac{4}{6}\right) = \frac{1}{5} - \frac{1}{6}, \qquad \frac{1}{4}\left(\frac{4}{7} - \frac{4}{8}\right) = \frac{1}{7} - \frac{1}{8};$$

$$\frac{1}{2^n}\left(\frac{2^n}{2^n + 2k - 1} - \frac{2^n}{2^n + 2k}\right) = \frac{1}{2^n + 2k - 1} - \frac{1}{2^n + 2k}, \quad k = 1, 2, \cdots, 2^{n-1}; \; n = 1, 2, \cdots .$$

$$\ln 2 = \int_1^2 \frac{dx}{x} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

—Mark Finkelstein

The Proof is in the Picture

Conclusion

- Strengthen Analytical skills
- Innovative and creative
- Transferable skills
- Visual Learners
- Visuals are not a replacement for proofs

Works Cited

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Q.E.D