Difference Equations

Introduction

1. Brief overview of the basics of difference equations

2. Study of a general epidemic model that utilizes generating functions

3. Research and findings of Union University influenza data using a difference equation model

What is a Difference Equation?

At its basics, a difference equation is a relation between consecutive elements of a sequence. Difference equations enable the mathematician to use a given set of values in order to determine the value of a function.

◄ | ►

An example would be in finding the value of an investment:

Let y(t) be the value of an investment after t quarters of a year. Our original investment will be $24, so y(0) = 24. If our interest rate is 1.75% per quarter, then y(t) satisfies the difference equation

$$\begin{aligned} y(t+1) &= y(t) + .0175y(t) \\ &= y(t)[1 + .0175] \\ &= (1.0175)y(t) \end{aligned}$$

for t = 0, 1, 2, … . Computing y recursively, we have

$$\begin{aligned} y(1) &= 24\,(1.0175) \\ y(2) &= 24\,(1.0175)^2, \\ &\;\;\vdots \\ y(t) &= 24\,(1.0175)^t\,. \end{aligned}$$

Difference Operator

Let y(t) be a function of a real or complex variable t. The "difference operator" Δ is defined by

$$\Delta y(t) = y(t+1) - y(t).$$

There are several rules, theorems, and properties that go along with the difference operator.

For instance:

-the higher order difference rule (similar to a 2nd derivative in differential calculus),

-the factorial function (a version of the power rule for solving finite differences)

◄ | ►

To show how simple and accessible a proof in difference calculus can be, an example is given for the theorem

Theorem: $\Delta a^t = (a - 1)a^t$.

Proof: $\Delta a^t = a^{t+1} - a^t$
$= a^t(a^1 - 1)$.

◀ | ▶

Summation

An “indefinite sum” of y(t), denoted $\sum y(t)$ is any function so that

$$\Delta\left(\sum y(t)\right) = y(t)$$

for all t in the domain of y.

◀ | ▶

Generating Function

Let $\{y_k\}$ be a sequence of constants. Suppose there is a function g(x) so that

$$g(x) = \sum_{k=0}^{\infty} y_k x^k$$

for all x in an open interval about 0. Then g is called the “generating function” for $\{y_k\}$

◄ | ►

Finding a Generating Function

Let $y_k = C^k$, for some constant C. To compute the generating function for $\{y_k\}$, the series must be summed,

$$\sum_{k==}^{\infty} C^k x^k = \sum_{k=0}^{\infty} (\text{Cx})^k$$

This can be recognized as a geometric series and the generating function is found.

$$\frac{1}{1-\text{Cx}} = \text{g(x)}$$

for $|\text{ Cx}| \prec 1$.

◂ | ▸

Epidemiology Model

◀ | ▶

The epidemic model investigated below is presented in Kelley and Peterson, pages 87 and 88 and Lauwerier, page 162.

x_n = the fraction of susceptible individuals in a certain population during the n^{th} day of an epidemic
A_k= a measure of how infectious the ill individuals are during the k^{th} day
ϵ = a small positive constant representing the carriers (those who can not get the disease but are able to spread it to others).

$$\log \frac{1}{x_{n+1}} = \sum_{k=0}^{n} (1 + \epsilon - x_{n-k}) A_k, \qquad (n \geq 0)$$

With substitutions and the utilization of exponent rules and properties of the natural log, this equation can be rewritten:

$$y_{n+1} = \sum_{k=0}^{n} (\epsilon + y_{n-k}) A_k.$$

where y_n is the fraction of people that have the disease.

◄ | ►

The method of generating functions can be applied here because of the form of the sum $\sum_{k=0}^{n} y_{n-k} A_k$, which is called a sum of "convolution type."

Now, a generating function Y(t) must be derived for $\{y_n\}$,

$$Y(t) = \sum_{n=0}^{\infty} y_n t^n,$$

and set

$$A(t) = \sum_{n=0}^{\infty} A_n t^{n+1}.$$

By multiplying the two power series and factoring, the product is

$$A(t)Y(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} y_{n-k} A_k\right) t^{n+1}.$$

Returning to the equation $y_{n+1} = \sum_{k=0}^{n} (\epsilon + y_{n-k}) A_k$ and distributing A_k as well as multiplying both sides of this difference equation by t^{n+1} and summing results in

$$\sum_{n=0}^{\infty} y_{n+1} t^{n+1} = \epsilon \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} A_k \right) t^{n+1} + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} A_k \, y_{n-k} \right) t^{n+1}.$$

By simplification, substitution, and factoring, a generating function for $\{y_n\}$ is derived:

$$\mathrm{Y(t)} = \frac{\epsilon A(t)}{(1-t)(1-A(t))}.$$

◄ | ►

In a few special cases, the sequence $\{y_n\}$ can be computed explicitly. The specific model that will be used with the Union University data is presented on pages 88 and 89 of the Kelley and Peterson text.

Conditions:

$A_k = c\alpha^k$, $0 < \alpha < 1$.
c is a constant that represents the infectiousness on day 0
α is the rate at which the infectiousness declines daily.

Set $A(t) = \frac{ct}{1-\alpha t}$ and $Y(t) = \frac{\epsilon ct}{(1-t)(1-\alpha t - ct)}$.

By partial fractions, simple algebra, and recognition of geometric series, the equation for Y(t) can be rewritten,

$$Y(t) = \left(\frac{\epsilon c}{1-(\alpha+c)}\right)\left[\sum_{n=0}^{\infty} t^n - \sum_{n=0}^{\infty} (\alpha + c)^n t^n\right].$$

So, for a single value of y_n, it is written

$$y_n = \left(\frac{\epsilon c}{1-(\alpha+c)}\right)[1 - (\alpha + c)^n].$$

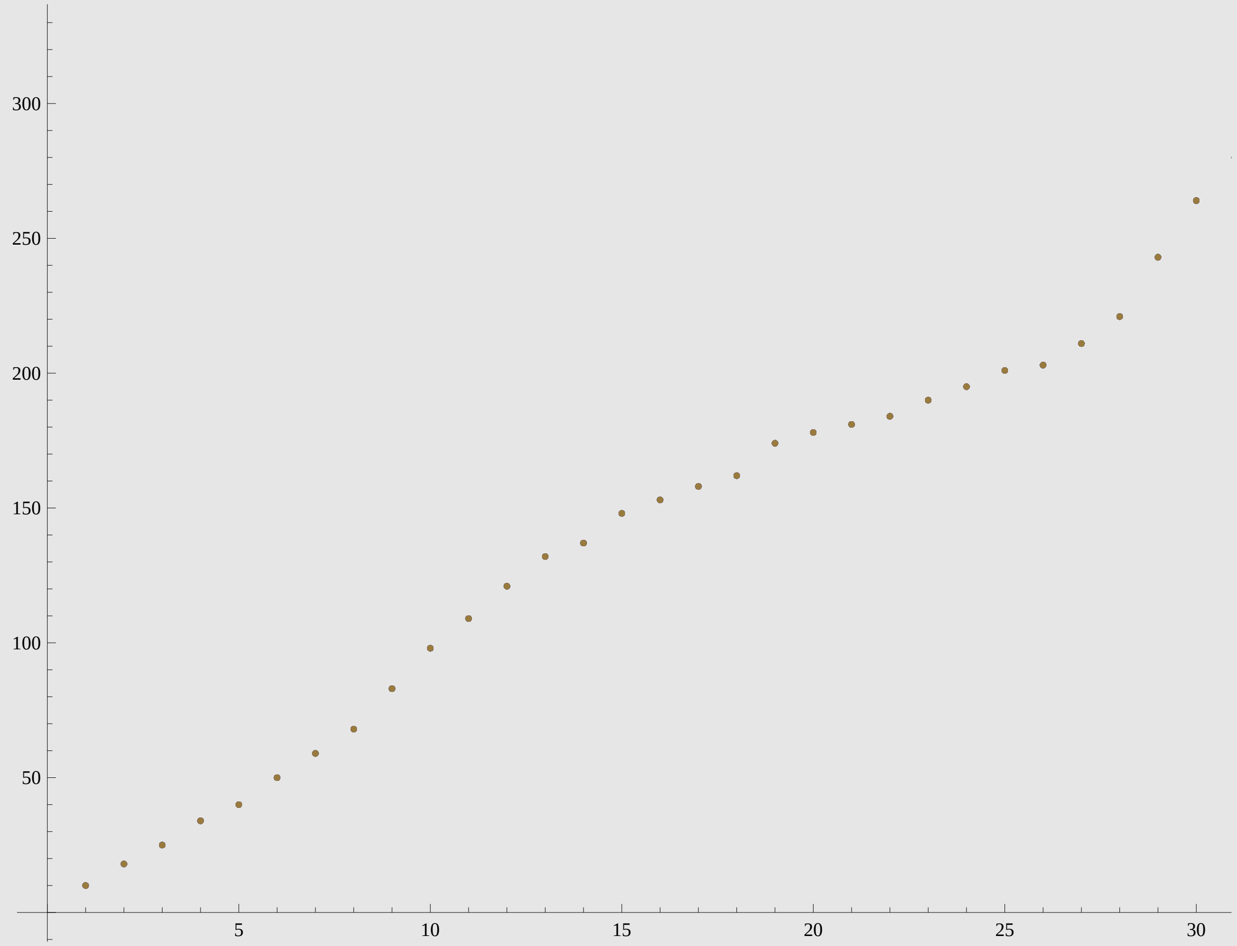
◀ | ▶

Union University and Influenza

◀ | ▶

```
uuprebreak := {10, 18, 25, 34, 40, 50, 59, 68,
  83, 98, 109, 121, 132, 137, 148, 153, 158, 162,
  174, 178, 181, 184, 190, 195, 201, 203, 211, 221,
  243, 264, 280, 285, 302, 304, 311, 318, 321, 326}
```

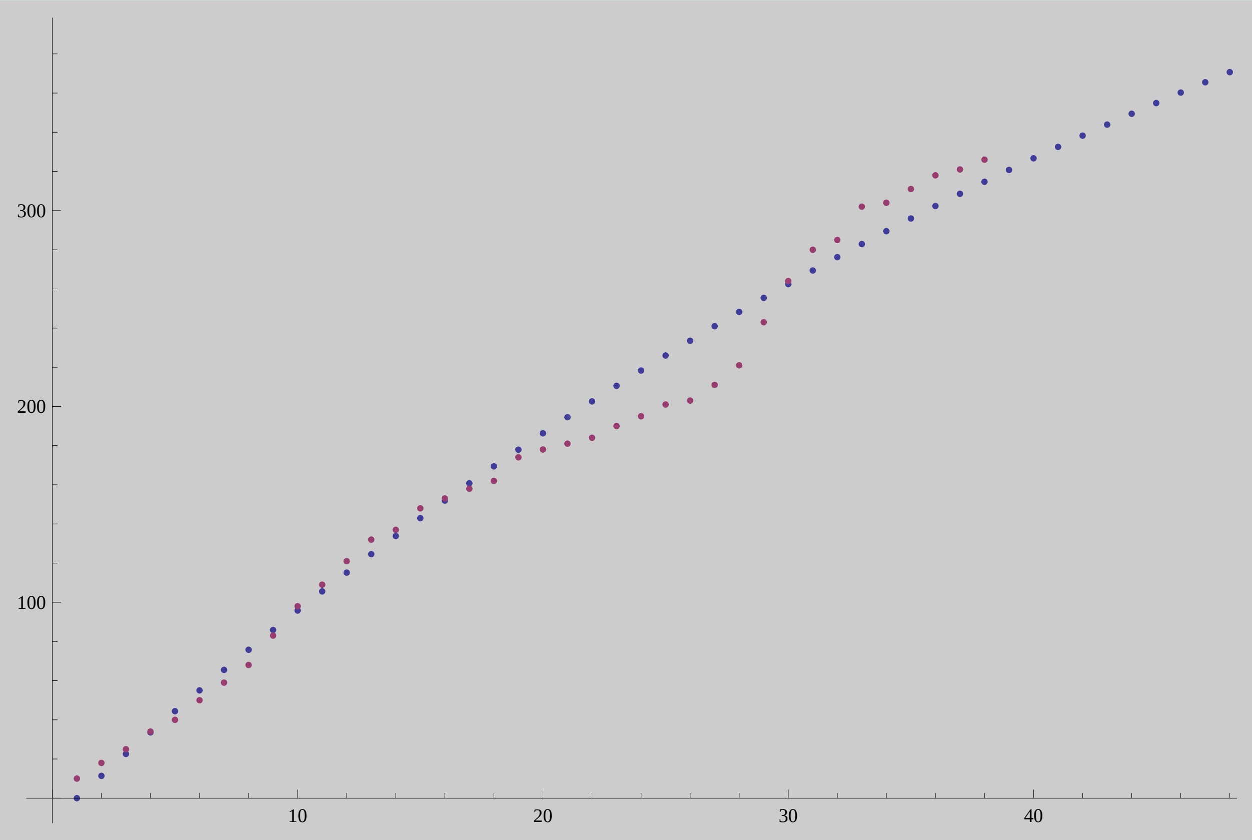
```
ListPlot[Table[uuprebreak, {n, 0, 40}]]
```



Graph the generating function with particular values of ϵ, α, and c.

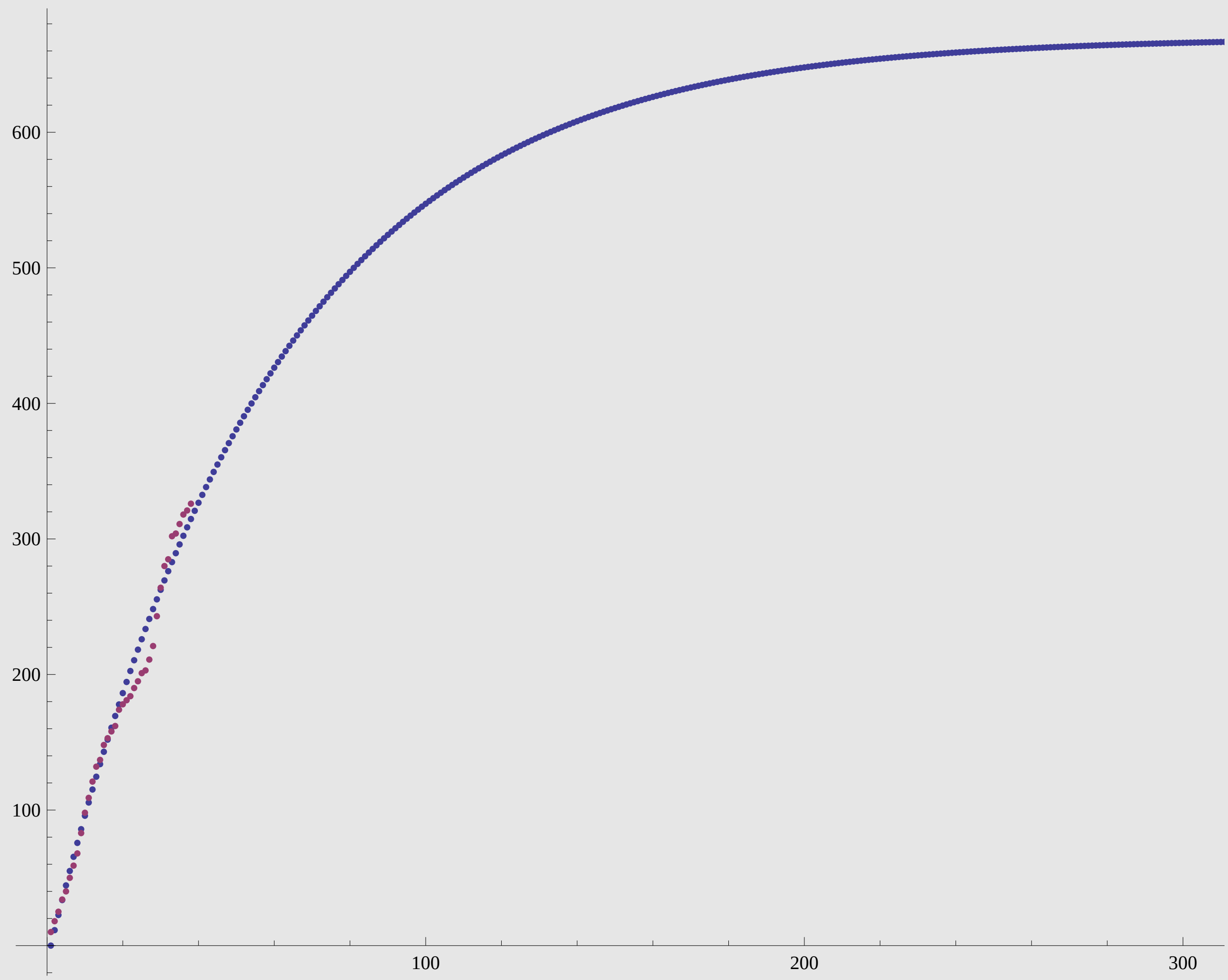
```
ϵ := .0088
α := .53
c := .453
```

```
ListPlot[
 {Table[2857 * (ϵ * c) / (1 - (α + c)) (1 - (α + c) ^n),
   {n, 0, 50}], uuprebreak}]
```



Expanding the domain shows when the graph will level off.

```
ListPlot[{Table[2857 * (ϵ * c) / (1 - (α + c)) (1 - (α + c)^n), {n, 0, 396}],
  uuprebreak}]
```

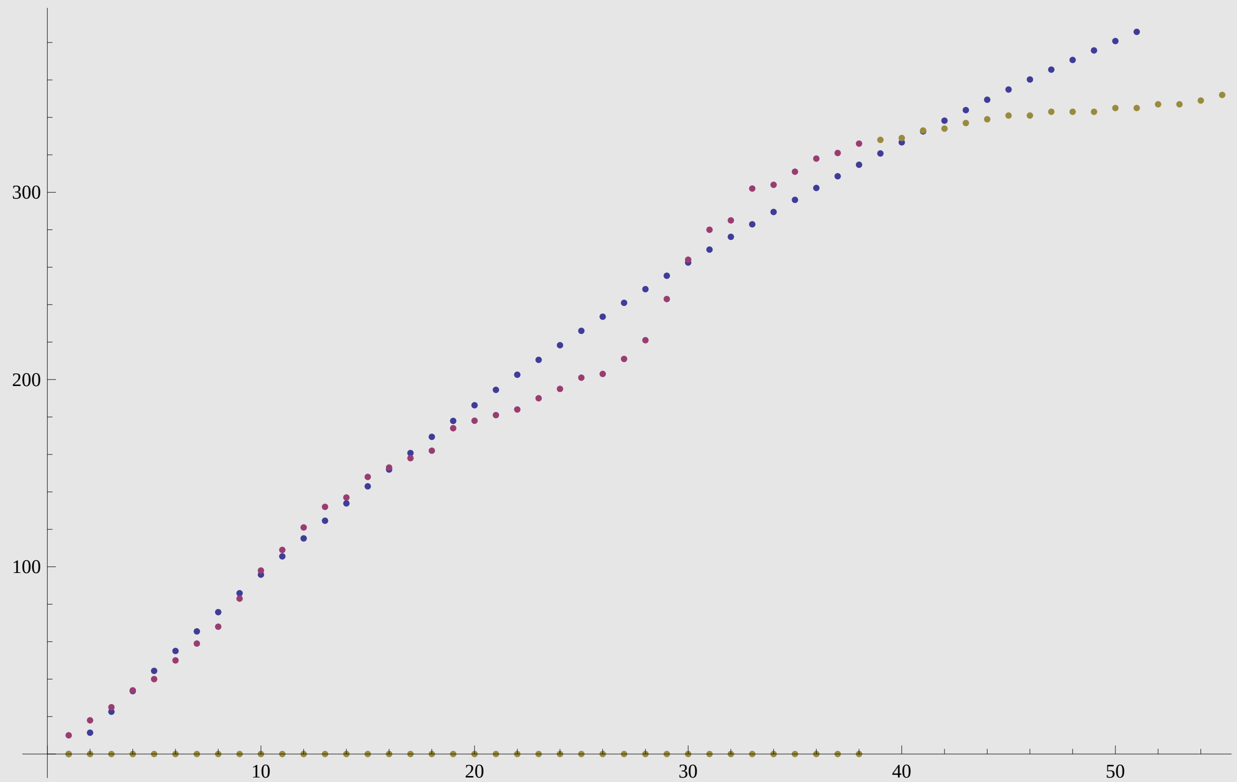


◄ | ►

Now, look at the post-fall break data and compare.

```
uupostbreak := {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 328, 329, 333, 334, 337, 339, 341, 341,
  343, 343, 343, 345, 345, 347, 347, 349, 352, 353}
```

```
ListPlot[
 {Table[2857 * (ϵ * c) / (1 - (α + c)) (1 - (α + c) ^n),
   {n, 0, 50}], uuprebreak, uupostbreak}]
```



Possible Conclusions to Research:

Natural trend

Extended break

Need more data

◄ | ►

References

H. Lauwerier, Mathematical Models of Epidemics, Math. Centrum, Amsterdam, 1981.

Kelley, Walter, and Peterson, Allan. Difference Equations: An Introduction with Applications. San Diego, CA: Academic Press, Inc, 1991.

Union University Health Services Director, Paul Mayer, provided the Union University flu outbreak data.

Acknowledgements

Dr. Lunsford for his work in the senior seminar class

Dr. Dawson for his patience and guidance in working through difference equations