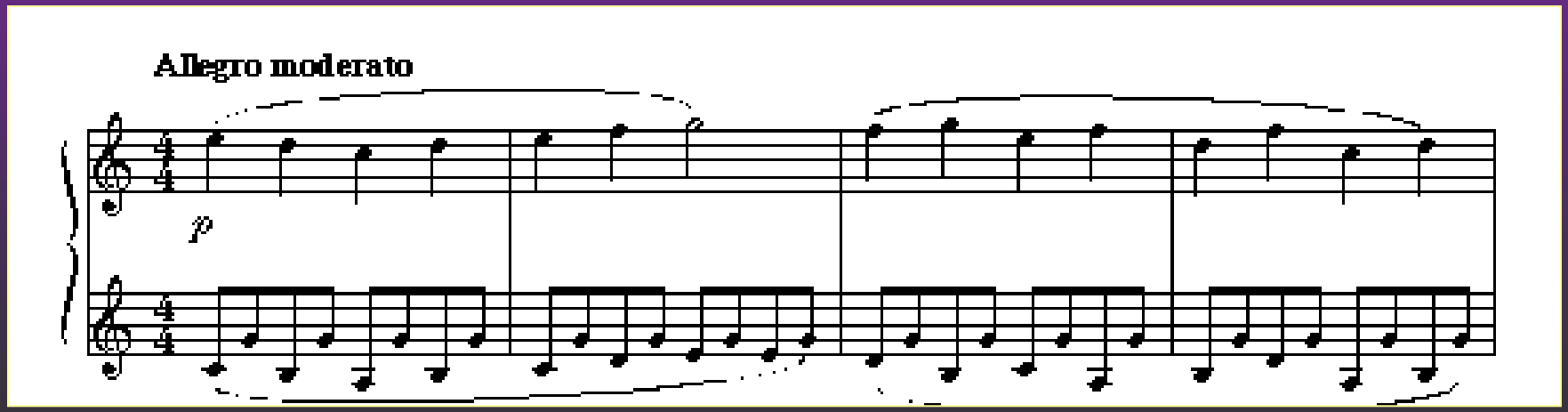
The Mathematics of Music

by

Lindsey Crain



Mathematics

"the queen of science"

Music

"the queen of art"

↓**Chaldeans, Egyptians, Babylonians, Chinese**

↓***PYTHAGOREANS***

↓**QUADRIVIUM:**

arithmetic

geometry

astronomy

MUSIC

↓**RENAISSANCE**

The Mathematics of:

Musical Sound

Musical Structure

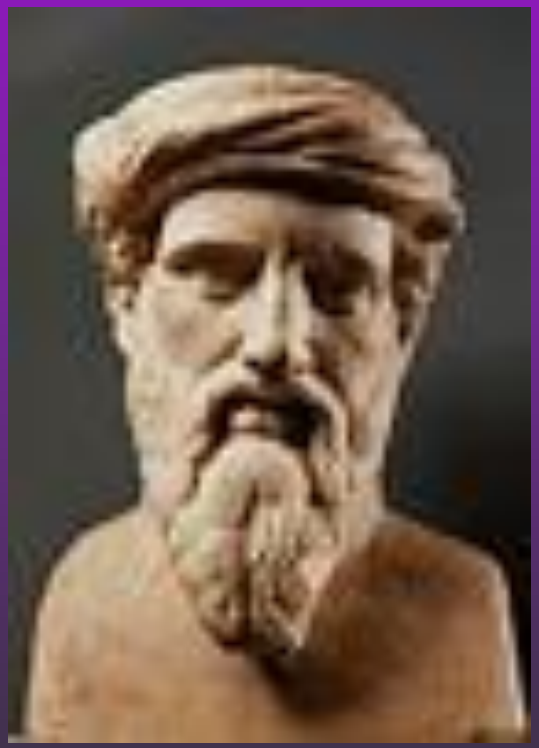
Musical Composition

The Mathematics of Sound

All of music is based on the physical concept of *vibrations*.

Early mathematicians who studied the mathematical basis of music:

Pythagoras



Euclid

Galileo

Descartes

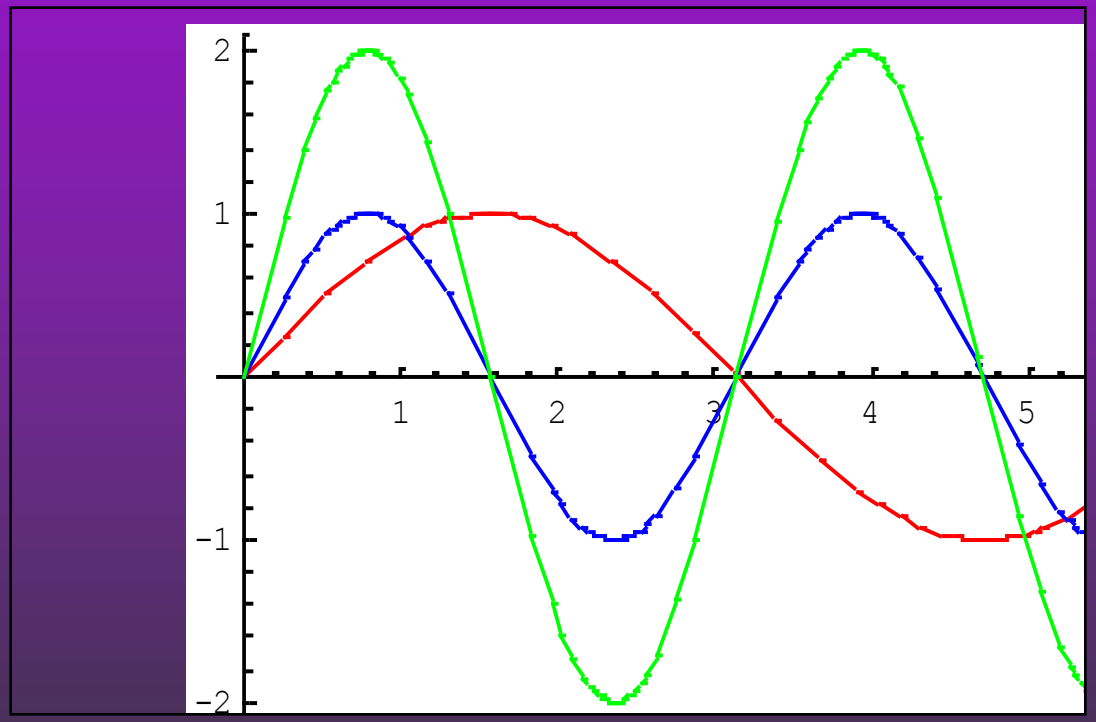
Further developments after the advent of modern science:

- Overtones

- Debate over the motion of the string

Fourier's theorem: "Every periodic phenomenon can be decomposed into a very large number of pure sine waves, or simple harmonic vibrations, whose frequencies are multiples of the frequency of the phenomenon under study."

Sine Curve

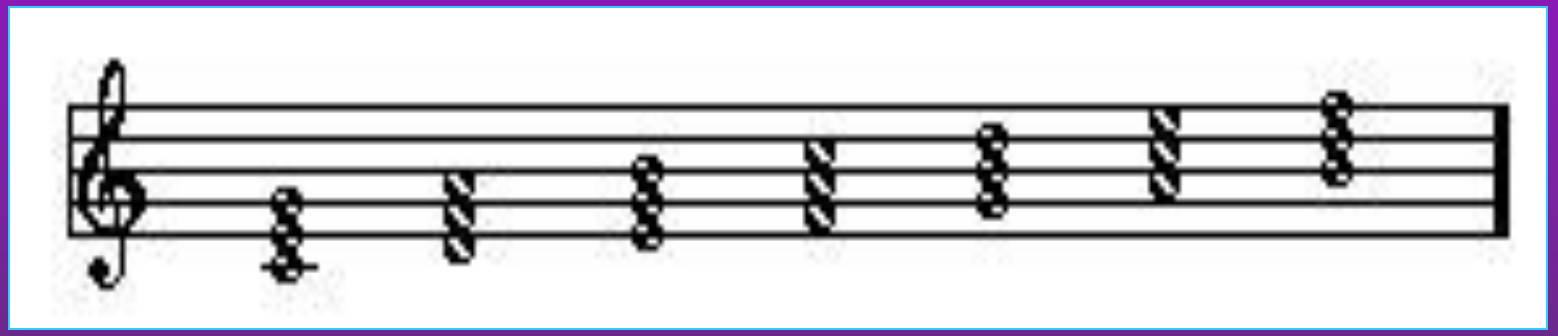


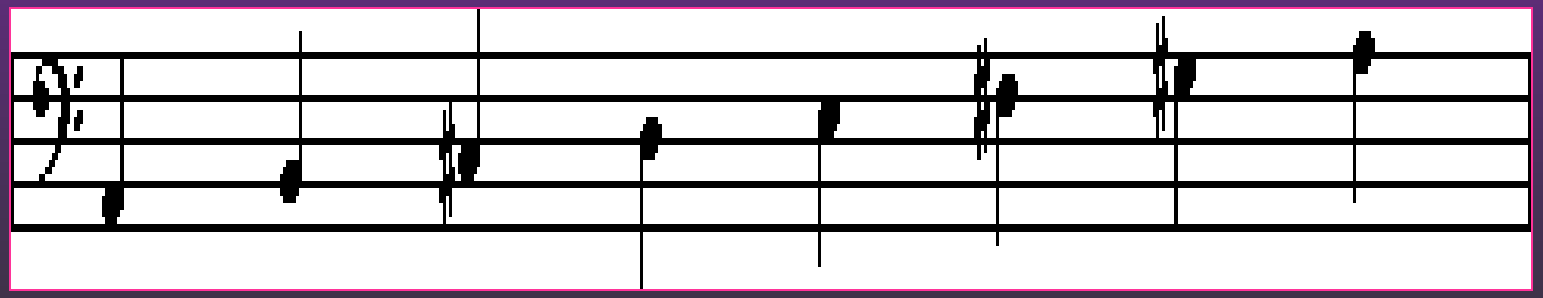
$y=\sin x$

$y=\sin(2x)$

$y=2\sin(2x)$

The Mathematics of Musical Structure





The Pythagoreans expressed the relation of the vibrating portion of a string to the whole string in ratios.

1:2 octave ***2:3 fifth***

↓Consonant sounds
(ratios of 1, 2, 3, or 4)

↓Dissonant sounds
(more complex ratios)

Development of the Scale

↓Pythagorean Scale

third harmonic

1	9/8	81/64	4/3	3/2	27/16	243/128	2
C	D	E	F	G	A	B	C

perfect fifth

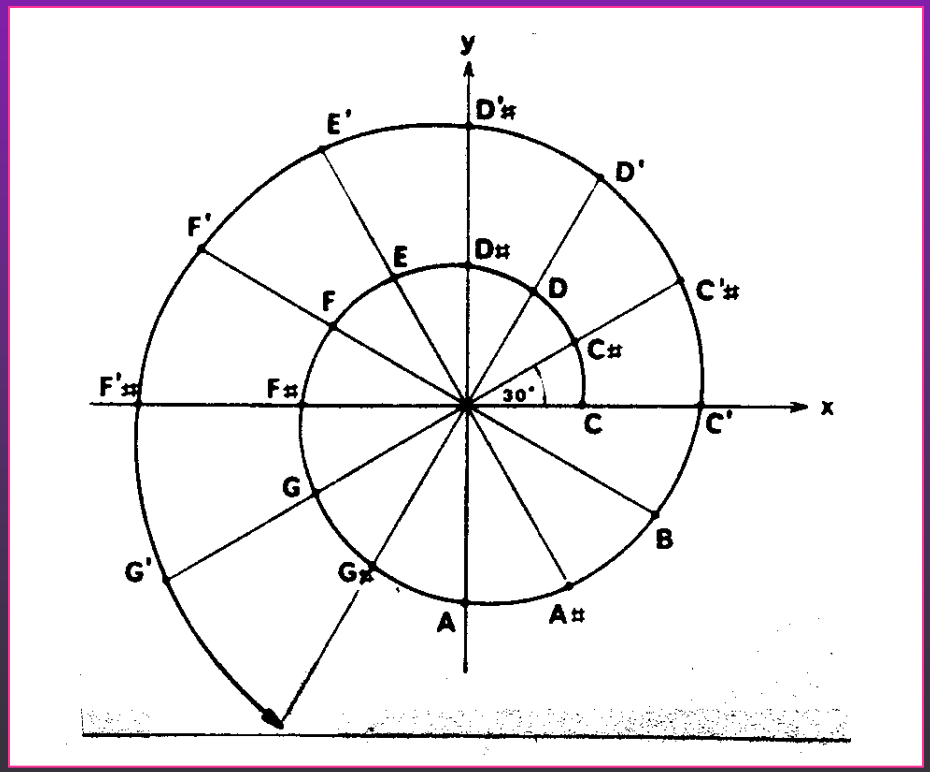
Just Intonation Scale

Equal Temperament Scale

Middle Eastern and Oriental Scales

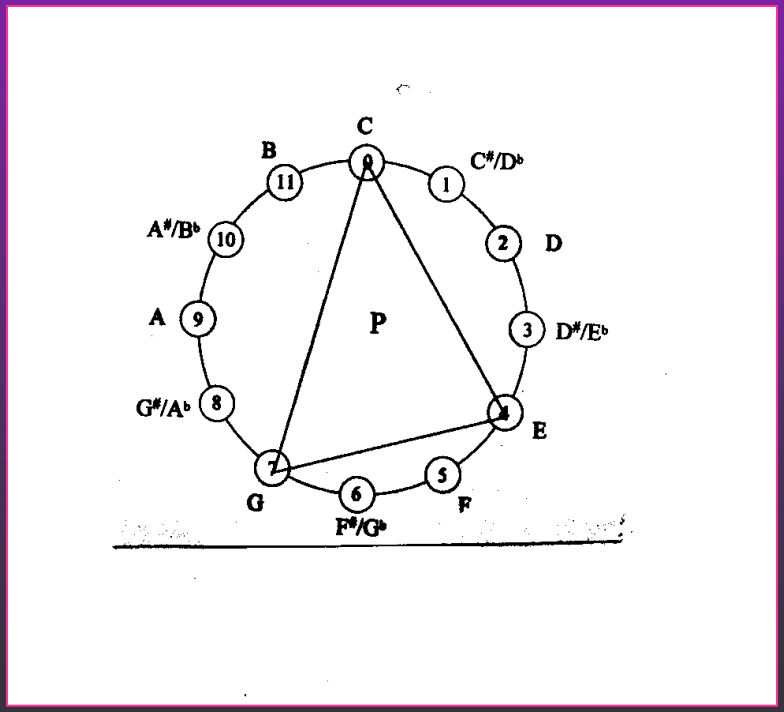
Logarithmic Spiral Representation of Pitch vs. Frequency

$r_n = r_0 q^n$ $q = e^{2\pi a}$



Z_{12} Analysis of Musical Chords

$Z_{12} = \{0, 1, . . ., 11\}$



Transposition:

$$T_n(P) = \{p_1+n, \ldots, p_k+n\}$$

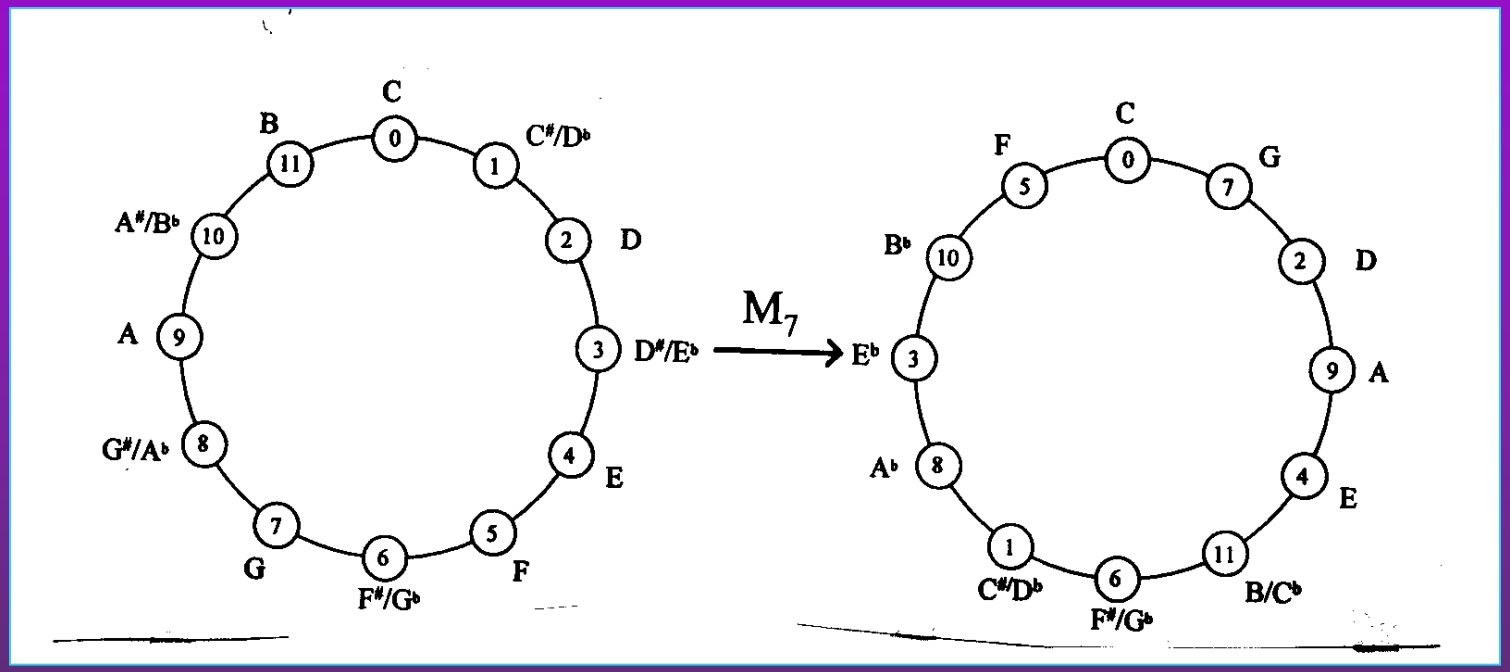
Inversion:

$$I_n(P) = \{-p_1+n, \ldots, -p_k+n\}$$

Twelve Tone Operators:

$$\{T_n, I_n \mid n=0, \ldots, 11\}$$

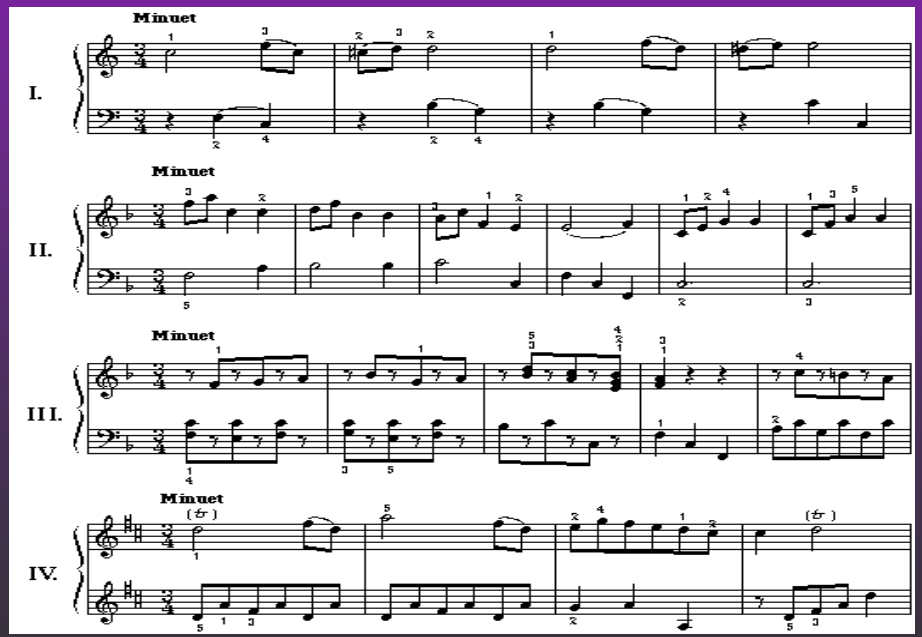
The circle of fifths transform



“closely related” keys

“remotely related” keys

The Mathematics of Musical Composition



- Similar thought processes
- Reflection in compositions

Both are aesthetically pleasing and logically challenging

↓The beauty of mathematics

Gauss's quick calculation: 1 + 2 + 3 + . . . +100 = 5050

Recognizing patterns

Contrast between size of task and simplicity of solution

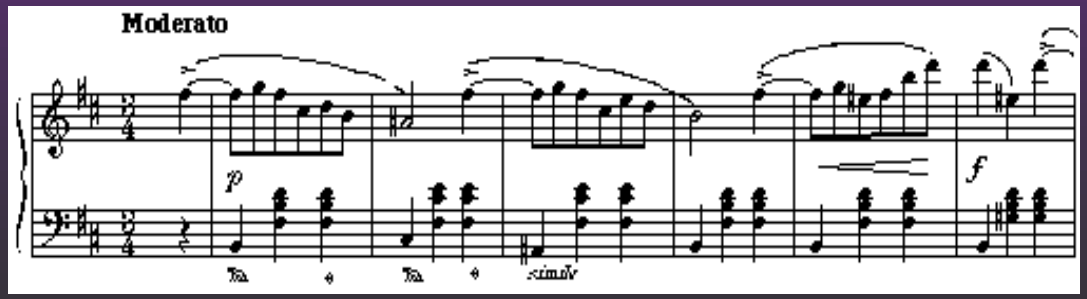
↓The logic behind music

Music exemplifies logical characteristics of mathematics.

Music also involves recognizing patterns.

Similarities in Reading and Understanding

↓Symbols □, ⇩, ↗



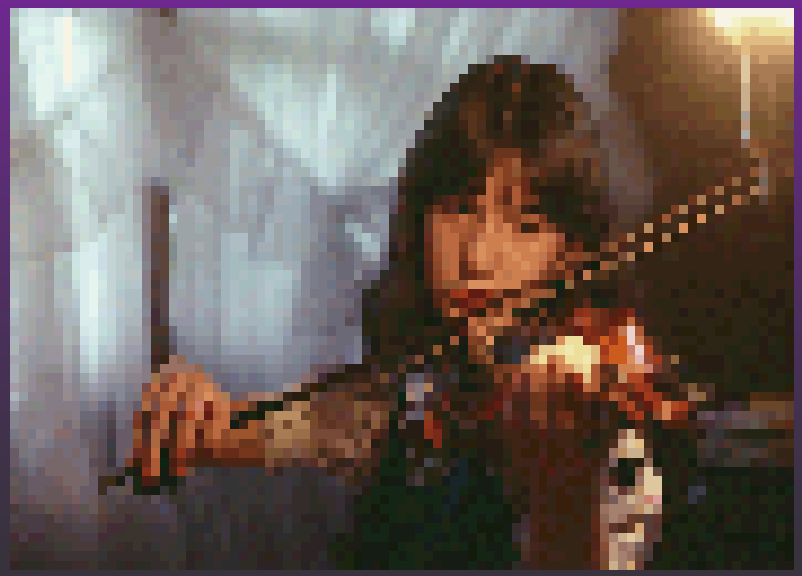
Similarities in Creation

↓Creation begins with new ideas--the selection of the most beautiful combination from an infinitude of possibilities

↓Once chosen the ideas are put into a critical ORDER

The effect of each field on one's ability to master the other

↓Talent in one often accompanies enthusiasm for the other.



Gordon Shaw discovers the *"Mozart Effect"*

↓**Neurological evidence**

↓**Physiological evidence**

↓**Statistical evidence**

Mathematical Ideas Hidden in Musical Compositions

MOZART

Fan of both music and mathematics

↓The Golden Section

The Golden Section in Mozart's Sonatas

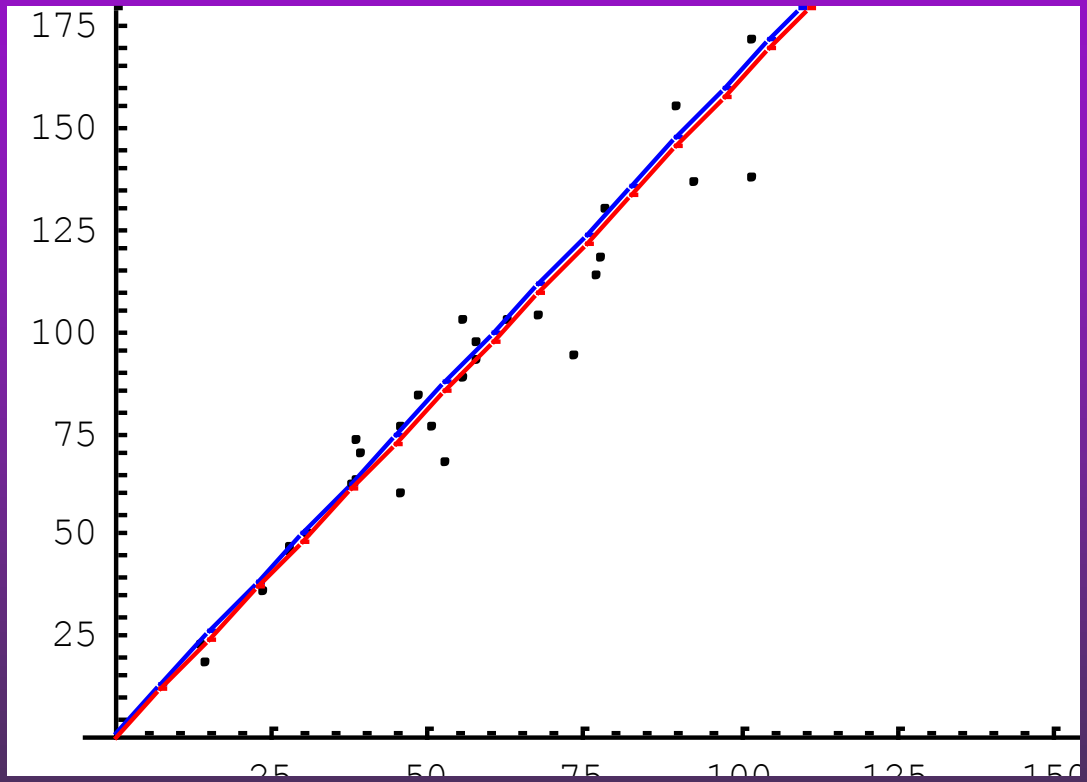
$$a/b = b/(a+b)$$

$$\nearrow = 1.6180$$

↓**Traditional form of a sonata:**

1. **Exposition (denoted *a*)**
2. **Development and Recapitulation (denoted *b*)**

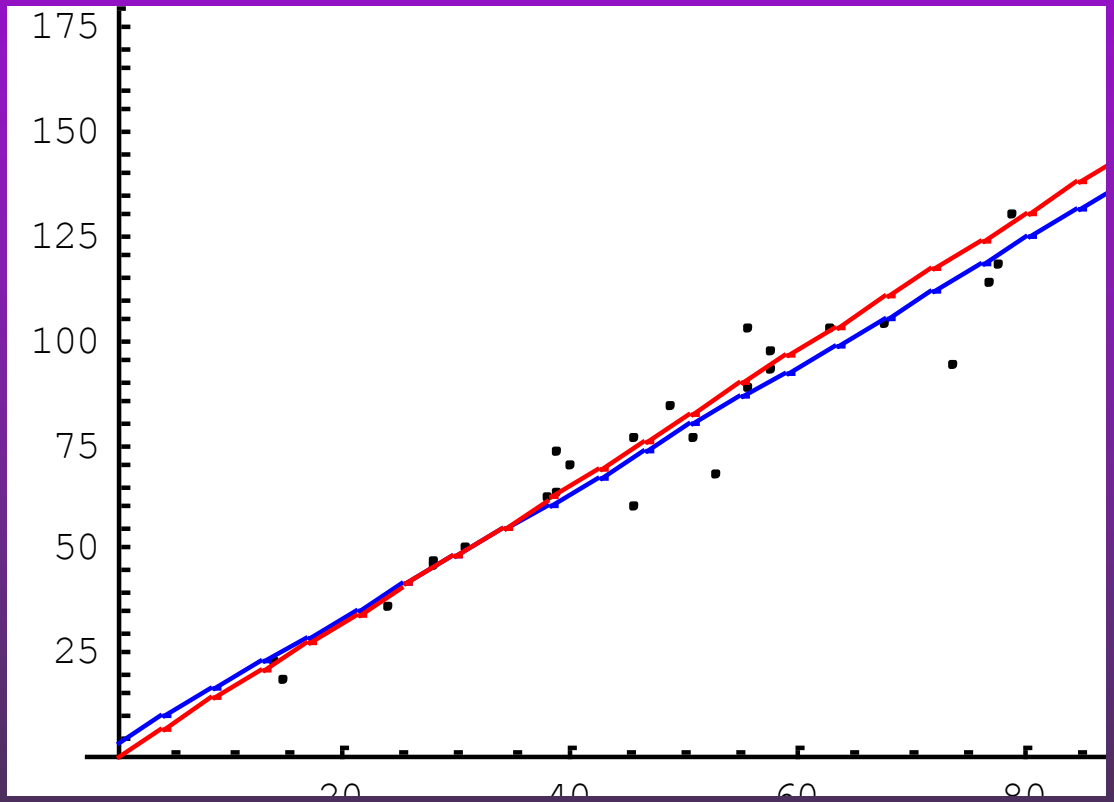
Plot of *b* against *a*+*b*



$y = x$ (red)

$y = -0.03241 + 0.6091x$ (blue)

Plot of *a* against *b*

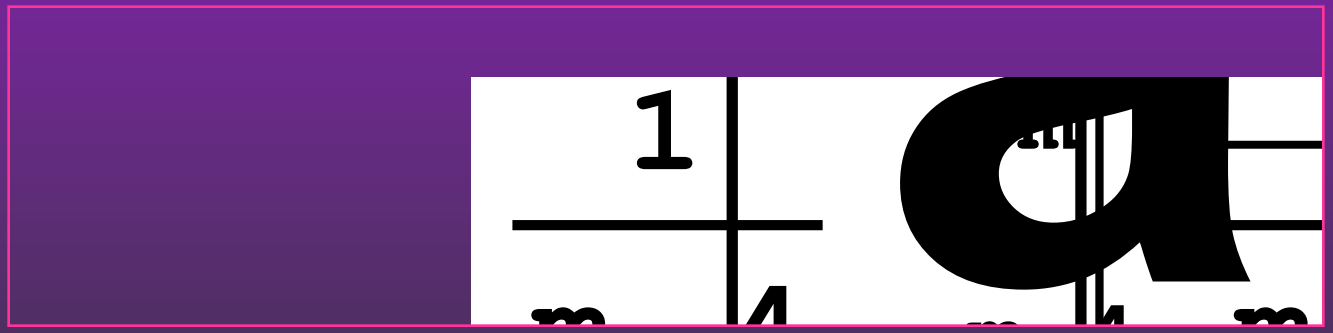


$y = x^{\nearrow}x$ (red)

$y = -0.03241 + 0.6091x$ (blue)

Possible values of *a/b*

If m/4 ≤ *a* ≤ m/2, then an estimate of the expected value of *a/b is*



0.6219

Some twentieth century composers who used mathematics:

John Cage

Milton Babbitt

Iannis Xenakis

Mathematics is “as useful to [the musician] as the learning of another language is to a poet.”

Igor Stravinsky

“Mathematics swims seductively just below the surface of music. It is a naiad gazing at the composer, seemingly within reach but actually unreachable.”

Eugene Helm

Einstein

Oak Ridge Scientists

Presenting . . .

“The Mathematical Mambo?”