Calculus in Business

By
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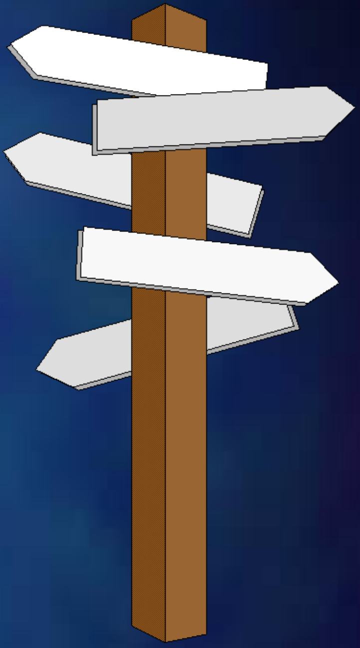
- Optimization
- Linear Programming
- Game Theory

Optimization
"The quest for the best"

Definition of goal equilibrium:

The equilibrium state is defined as the optimum position for a given economic unit and in which the said economic unit will be deliberately striving for attainment of that equilibrium.

- There are always many alternatives when any kind of an economic project is to be carried out.
- One or more will however be more desirable.



Most common criterion in economics

- Maximizing
- Minimizing

Definition of an objective function:

A function whose dependent variable represents the object of maximization or minimization and in which the set of independent variables indicates the objects whose magnitudes can be chosen with a view to optimizing.

If the first derivative of $f(x)$ at a point $x = x_0$ is $f'(x_0) = 0$ then the value of the function at this point will be :

- a relative maximum
- a relative minimum
- neither

Concept of the second derivative

and higher orders derivative

$$\frac{d^n y}{dx^n}$$

If $f'(x_0) = 0$ then the value of the function at that point will be

-a relative maximum if $f''(x_0) < 0$

-a relative minimum if $f''(x_0) > 0$

A firm must choose the output level

such that MC=MR

R=R(Q) total revenue function

C=C(Q) total cost function

Objective function

$$\Pi = \Pi(Q) = R(Q) - C(Q)$$

First condition for a maximum

$$\frac{d\pi}{dQ} = 0$$

$$\frac{d\pi}{dQ} = \pi'(Q) = R'(Q) - C'(Q) = 0$$

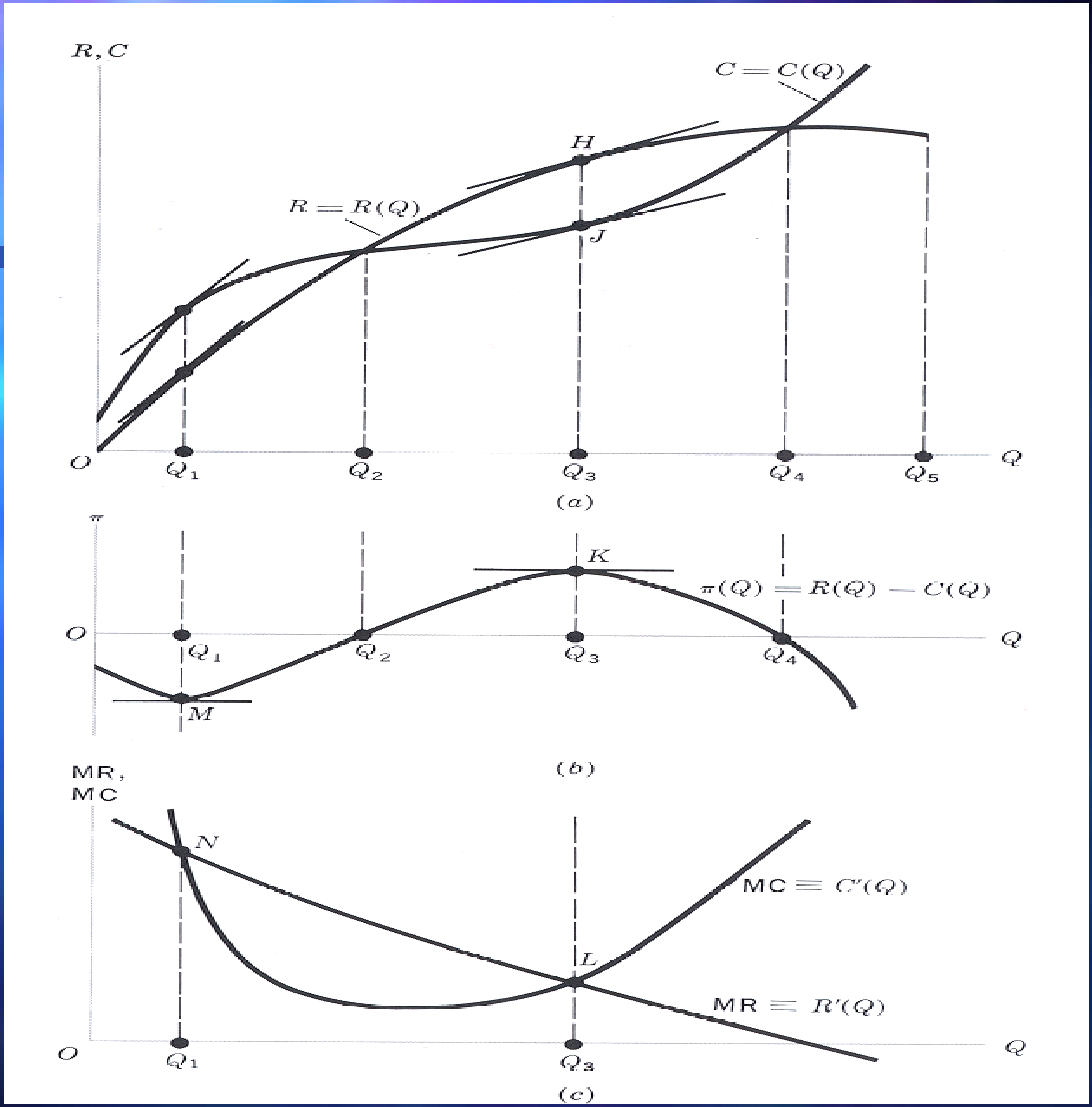
iff $R'(Q) = C'(Q)$

or $MR = MC$

Second condition

$$\frac{d^2\pi}{dQ^2} = \pi''(Q) = R''(Q) - C''(Q) < 0$$

iff $$R''(Q) < C''(Q)$$



R, C
C = C(Q)
H
R = R(Q)
J
O
Q₁
Q₂
Q₃
Q₄
Q₅
Q
(a)
π
K
π(Q) ≡ R(Q) − C(Q)
O
Q₁
Q₂
Q₃
Q₄
Q
M
(b)
MR,
MC
N
MC ≡ C′(Q)
L
MR ≡ R′(Q)
O
Q₁
Q₃
Q
(c)

An objective function for a multiproduct firm

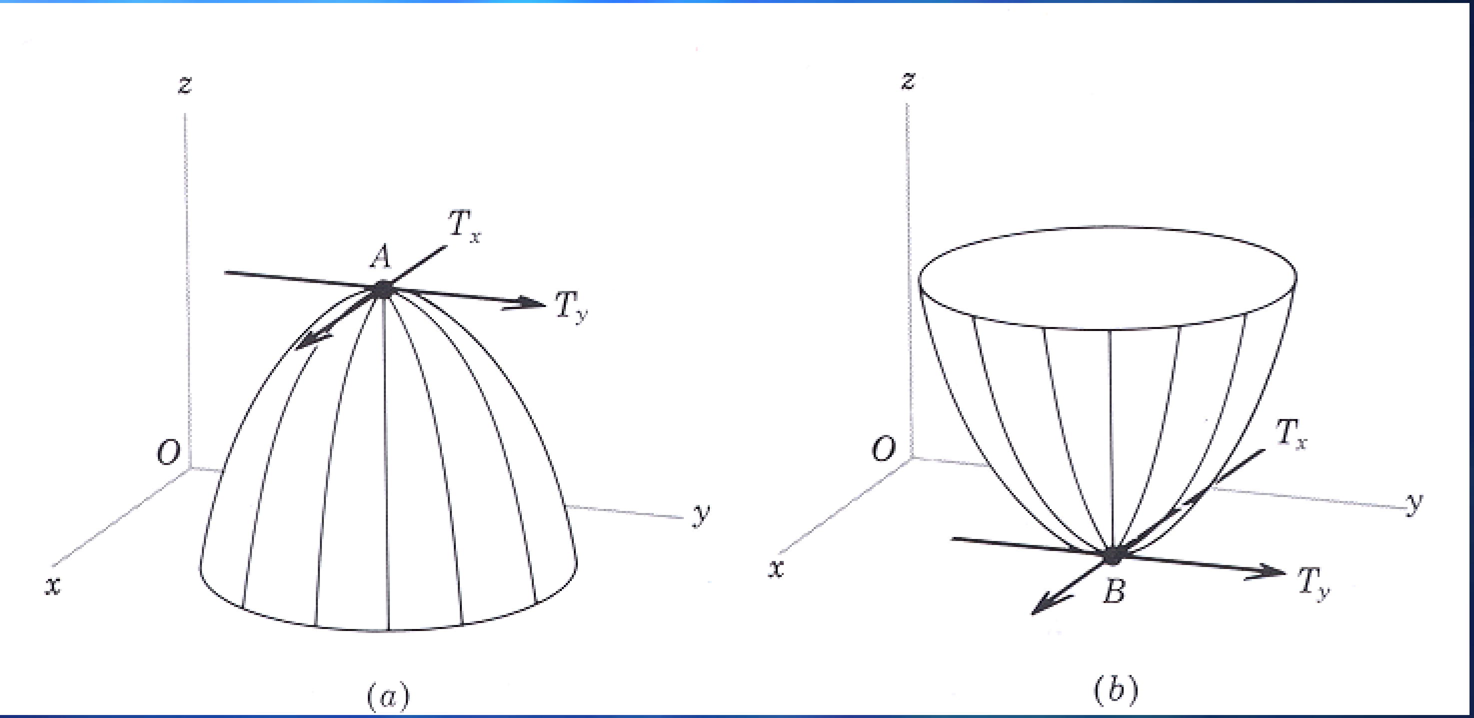
$$z = f(x, y)$$

$$\frac{\partial^2 z}{\partial x^2}$$

$$f_x \equiv \frac{\partial_z}{\partial_x}, \ f_y \equiv \frac{\partial_z}{\partial_y}$$

$$f_{xy} = \frac{\partial^2{}_z}{\partial_x \partial_y} \equiv \frac{\partial}{\partial_x}\left(\frac{\partial_z}{\partial_y}\right)$$

$$f_{yx} = \frac{\partial^2{}_z}{\partial_y \partial_x} \equiv \frac{\partial}{\partial_y}\left(\frac{\partial_z}{\partial_x}\right)$$


z
T_x
A
T_y
O
x
y
(a)
z
O
T_x
x
y
B
T_y
(b)

For a maximum

$$f_x = f_y = 0$$

$$f_{xx}, f_{yy} < 0, f_{xx} f_{yy} > f_{xy}^{\ 2}$$

For a minimum

$$f_x = f_y = 0$$

$$f_{xx}, f_{yy} > 0, f_{xx} f_{yy} > f_{xy}^{\ 2}$$

Example

$$R = P_{10}Q_1 + P_{20}Q_2$$

$$C = 2Q_1^{\,2} + Q_1Q_2 + 2Q_2^{\,2}$$

Note:

$$\frac{\partial C}{\partial Q_1} = 4Q_1 + Q_2$$

Profit function:

$$\pi = R - C = P_{10}Q_1 + P_{20}Q_2 - 2Q_1^2 - Q_1Q_2 - 2Q_2^2$$

$$\frac{\partial \pi}{\partial Q_1} = P_{10} - 4Q_1 - Q_2$$

$$\frac{\partial \pi}{\partial Q_2} = P_{20} - Q_1 - 4Q_2$$

$$4Q_1 + Q_2 = P_{10}$$

$$Q_1 + 4Q_2 = P_{20}$$

Solutions:

$$\bar{Q}_1 = \frac{4P_{10} - P_{20}}{15}, \bar{Q}_2 = \frac{4P_{20} - P_{10}}{15}$$

Second condition:

Hessian

$$|H| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ -1 & -4 \end{vmatrix}$$

$$|H_1| = -4 < 0$$

$$|H_2| = 15 > 0$$

Linear Programming

Technique that allows decision makers to solve maximization and minimization problems where there are certain constraints that limit what can be done.

The objective function is a linear function of the variables to be determined. The values of these variables must satisfy certain constraints, which are in the form of inequalities.

Production planning example:

Profit pair batch of cotton cloth finished is $1.00 with process 1, $0.90 with process 2, $1.10 with process 3.

Process 1 uses 3 machine-hours of finishing capacity per batch of cotton cloth processed, process 2 uses 2.50 machine-hours and process 3 uses 5.25 machine-hours.

Process 1 uses 0.40 hours of labor per batch of cotton cloth processed, process 2 uses 0.50 hours, and process 3 uses 0.35 hours.

6,000 machine-hours per week
600 hours per week

Problem: maximize

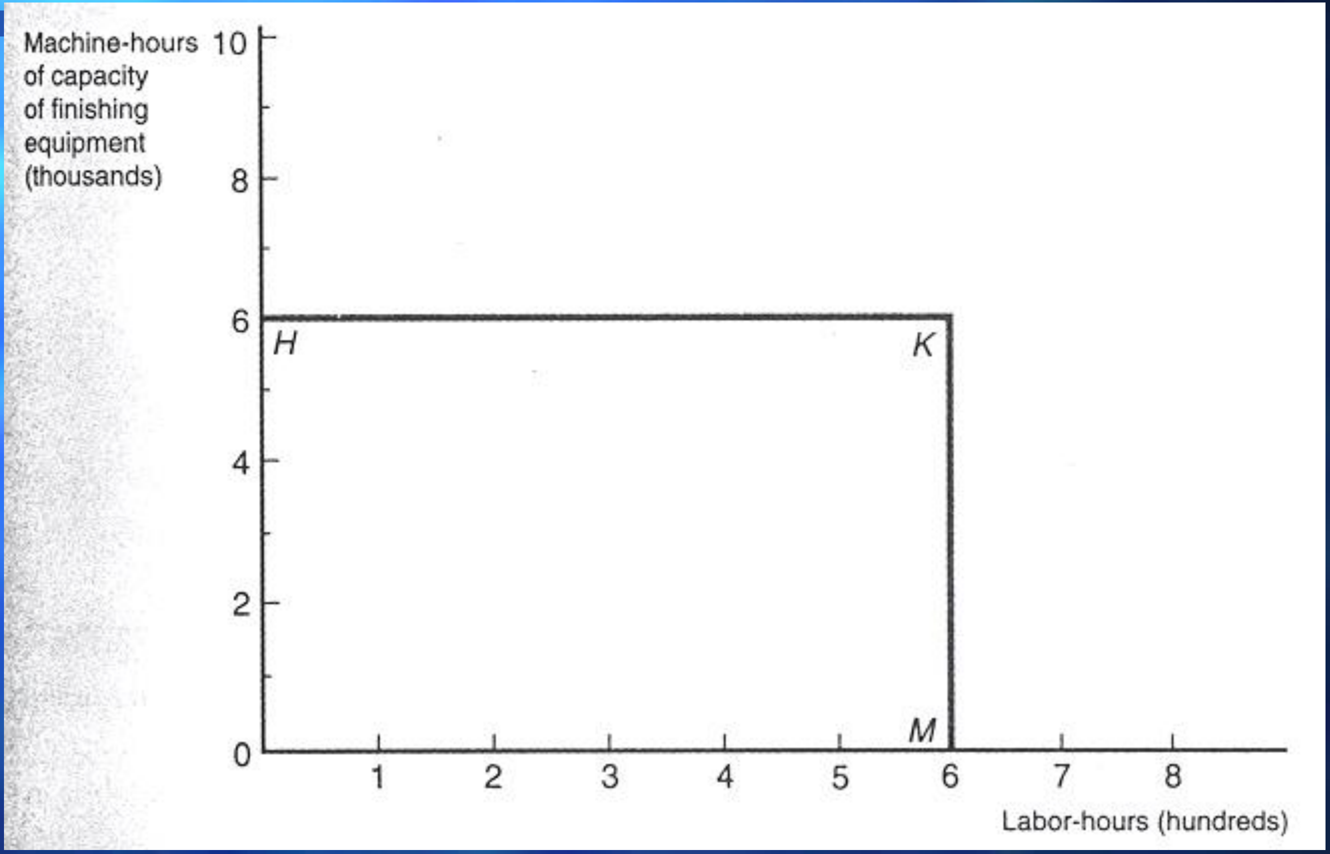
$$\pi = 1.00Q_1 + 0.90Q_2 + 1.10Q_3$$

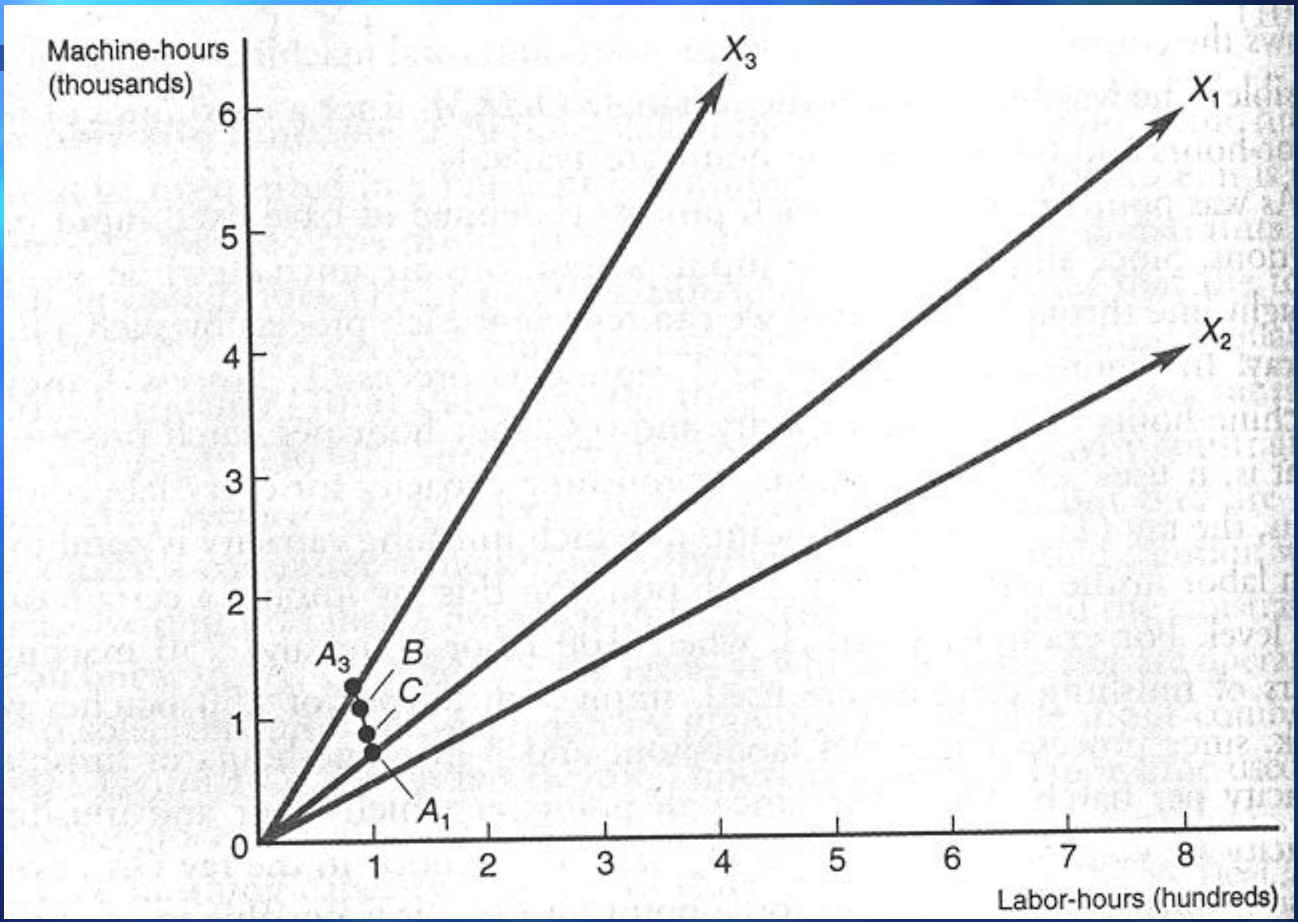
Constraints:

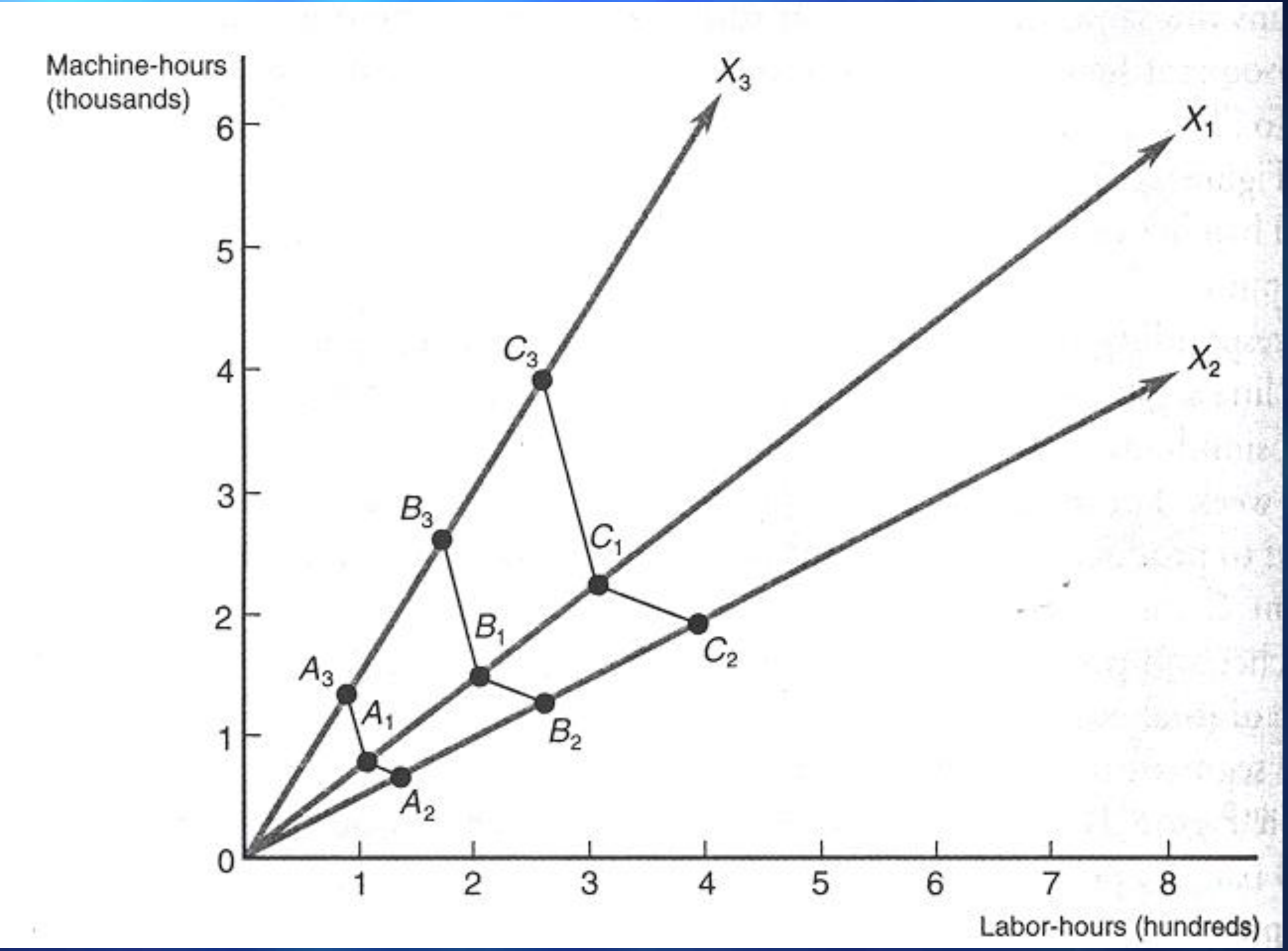
$$3Q_1 + 2.50Q_2 + 5.25Q_3 \le 6,000$$

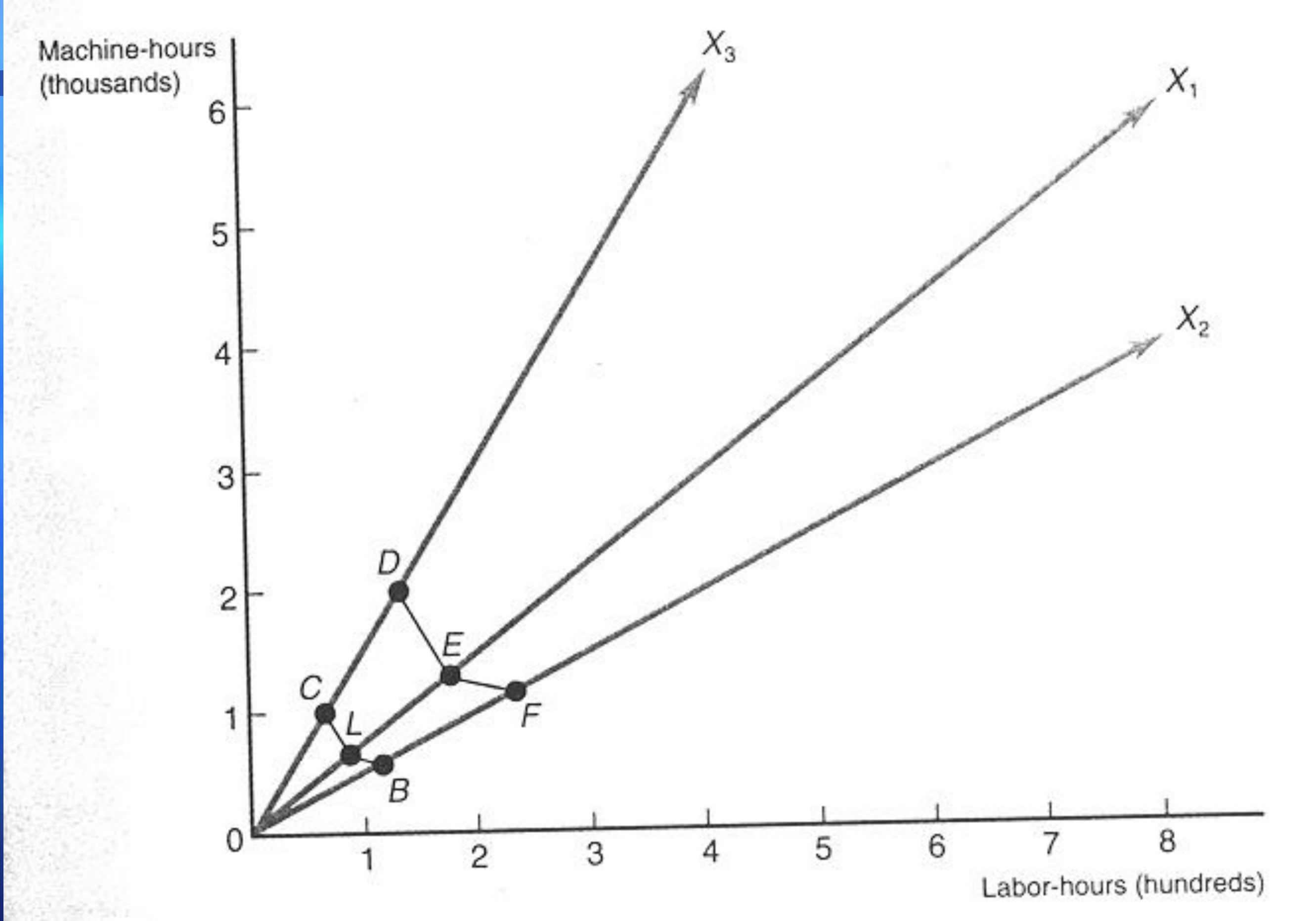
$$0.40Q_1 + 0.50Q_2 + 0.35Q_3 \le 600$$

$$Q_1, Q_2, Q_3 \ge 0$$

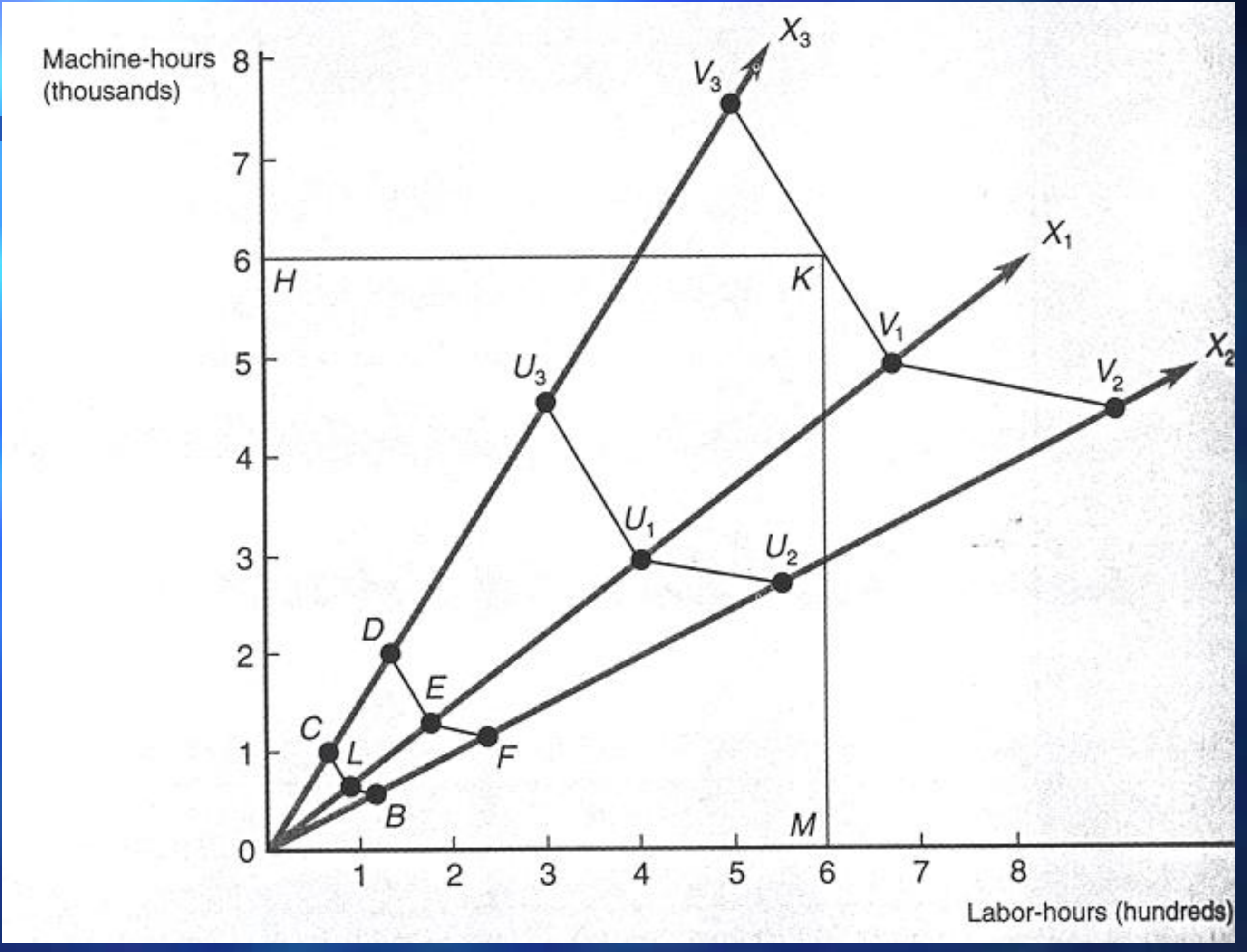

Machine-hours of capacity of finishing equipment (thousands)
10
8
6
4
2
0
H
K
M
1
2
3
4
5
6
7
8
Labor-hours (hundreds)


Machine-hours
(thousands)
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
Labor-hours (hundreds)
X3
X1
X2
A3
B
C
A1


Machine-hours (thousands)
Labor-hours (hundreds)
0
1
2
3
4
5
6
7
8
X_1
X_2
X_3
A_1
A_2
A_3
B_1
B_2
B_3
C_1
C_2
C_3



Machine-hours
(thousands)
X_3
X_1
X_2
6
5
4
3
2
1
0
D
E
C
L
F
B
1
2
3
4
5
6
7
8
Labor-hours (hundreds)



Machine-hours (thousands)
Labor-hours (hundreds)
0
1
2
3
4
5
6
7
8
H
K
M
C
L
B
D
E
F
U3
U1
U2
V3
V1
V2
X3
X1
X2

In this kind of problem the optimal solution will generally entail the use of no more processes than there are constraints.

$$3Q_1 + 5.25Q_3 = 6{,}000$$

$$0.40Q_1 + 0.35Q_3 = 600$$

$$Q_3 = 571.4$$

$$Q_1 = 1{,}000$$

Game Theory

A game is a competitive situation in which two or more persons pursue their own interests and no person can dictate the outcome

- The rules of the game
- A strategy
- A player’s payoff

Possible strategies for Allied	Possible strategies for Barkley	
	1	2
A	Allied's profit: $3 million Barkley's profit: $4 million	Allied's profit: $2 million Barkley's profit: $3 million
B	Allied's profit: $4 million Barkley's profit: $3 million	Allied's profit: $3 million Barkley's profit: $2 million

Possible strategies for Allied	Possible strategies for Barkley	
	1	2
A	Allied's profit: $3 million Barkley's profit: $4 million	Allied's profit: $2 million Barkley's profit: $3 million
B	Allied's profit: $4 million Barkley's profit: $3 million	Allied's profit: $3 million Barkley's profit: $4 million

A Nash equilibrium is a set of strategies such that each player believes that it is doing the best it can given the strategy of the other player(s).

Selection in Dynamic Entry Games

The Model

$$i \in \{1,2\}$$

$\Gamma_N(i)$ the N-stage game in which firm i is the initial leader.

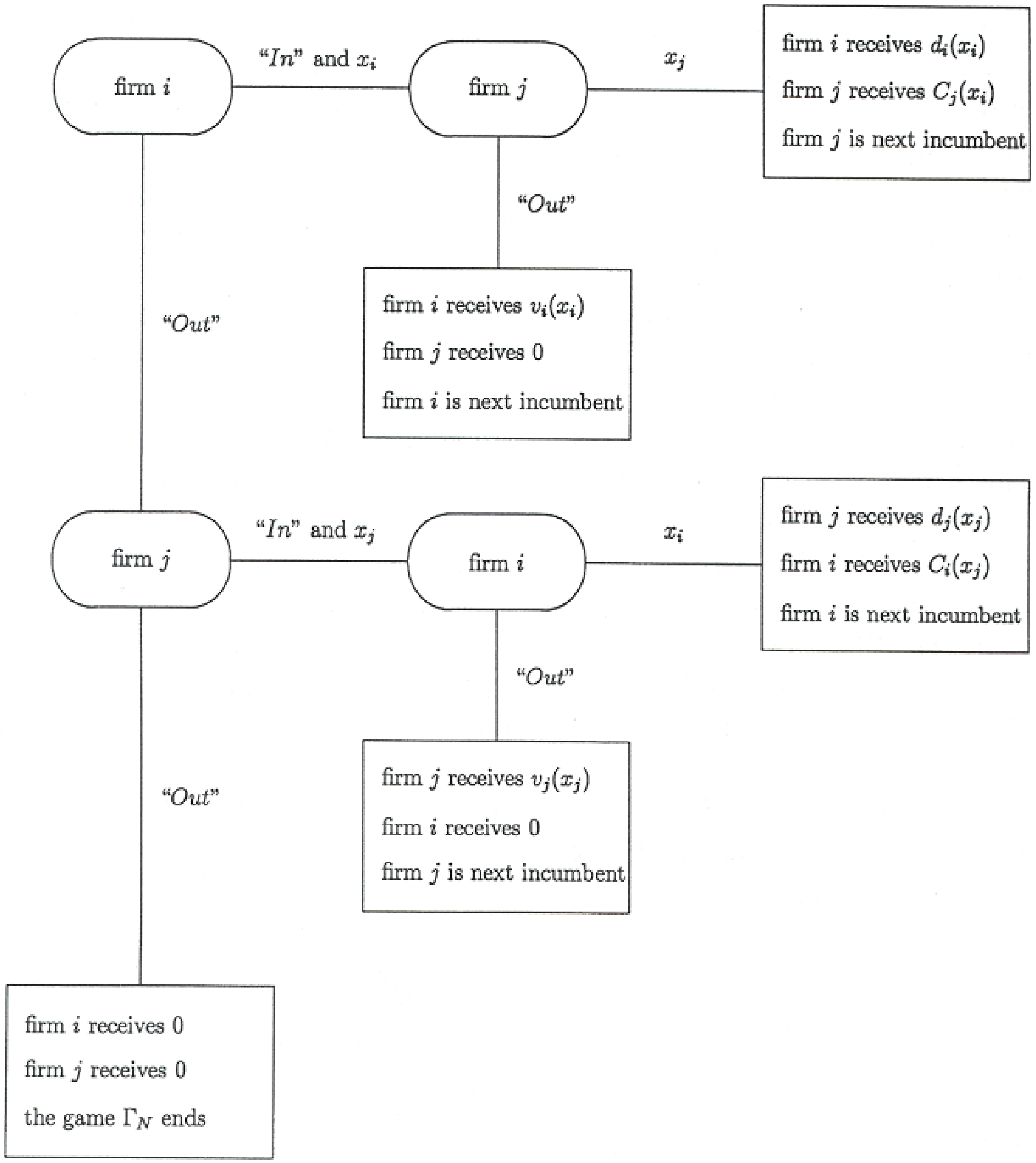
Game $G(i)$ is played as follows. First, firm i chooses a policy $x_i \in X$, where X is an interval in $\mathbf{R}$. Second, firm j (with $j \neq i$) chooses a policy $x_j \in \{Out, In\}$. The payoffs v_i and v_j are as follows:

$$v_i(x_i, x_j) = \begin{cases} v_i(x_i) & \text{if } x_j = Out \\ d_i(x_i) < 0 & \text{if } x_j = In; \end{cases}$$

$$v_j(x_i, x_j) = \begin{cases} 0 & \text{if } x_j = Out \\ -C_j(x_i) & \text{if } x_j = In. \end{cases}$$

If $G(i)$ has been played in stage n, the leader in the next stage, i.e. stage $(n - 1)$, is

$$\begin{array}{ll} \text{firm } i & \text{if } x_j = Out \\ \text{firm } j & \text{if } x_j = In. \end{array}$$


firm i
"In" and x_i
firm j
x_j
firm i receives $d_i(x_i)$
firm j receives $C_j(x_i)$
firm j is next incumbent
"Out"
firm i receives $v_i(x_i)$
firm j receives 0
firm i is next incumbent
"Out"
firm j
"In" and x_j
firm i
x_i
firm j receives $d_j(x_j)$
firm i receives $C_i(x_j)$
firm i is next incumbent
"Out"
firm j receives $v_j(x_j)$
firm i receives 0
firm j is next incumbent
"Out"
firm i receives 0
firm j receives 0
the game Γ_N ends

Assumption A: the functions v_1, v_2, C_1, C_2

are continuous on X.

Assumption B: the functions v_1, v_2

are strictly increasing on X.

Assumption C: the functions C_1, C_2

are strictly decreasing on X.

Assumption D: the functions $(C_1 + v_1), (C_2 + v_2)$

are nonincreasing on X.

Symmetric Firms

$$v_1 = v_2 = v_3$$

$$C_1 = C_2 = C_3$$

$$v(x^l) = 0$$

Average cost pricing

$$C(x^L) = 0$$

The firm's OEPP

$$x^1 = x^L$$

$$C(x^{n+1}) = \sum_{n=1}^{n} v(x^k)$$ for

$$n = 1, \ldots, N-1$$

In the unique perfect equilibrium of any symmetric entry game $\Gamma_N(i)$ with $i \in \{1,2\}$ firm i maintains with the OEPP $\{x^N, x^{N-1}, \ldots, x^1\}$

Total rent: $\sum_{n=1}^{N} v(x^n)$

Firm i plays "In" and $x_i \le x^{N+1}$

Firm j plays "In" its payoff is

$$-C(x_i) + \sum_{n=1}^{N} v(x^n) \le 0$$

i 's payoff is $v(x_i) + \sum_{n=1}^{N} v(x^n)$

$x_i \le x^{N+1}$, payoff maximized at

$x^{N+1}, v(x^{N+1}) > 0$

Firm i plays "In" and $x_i > x^{N+1}$

Firm j enters, its total profit is

$$-C(x_i) + \sum_{n=1}^{N} v(x^n) > 0$$

$$d_i(x_i) < 0$$

If firm i plays “Out”, G(j) is played j plays “In” and x^{N+1} and its OEPP

$$v(x^{N+1}) + \sum_{n=1}^{N} v(x^{n}) > 0$$

Optimal choice for firm i

“In” and x^{N+1}

Merci beaucoup