

## Highway Relativity

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Have you ever been driving just below the speed limit on the highway, and felt that most of the cars on the road were speeding? You may have been correct, or it may have been an illusion.

We will assume that on the highway you primarily observe cars that are immediately around your vehicle. Then, for traffic traveling at a variety of speeds, most of the cars that you observe pass you, or you pass them. Now, if you are traveling a little slower than average, more pass you. We shall see that even if you accurately judge the speed of each individual car that you see, when you travel slower than the average speed of traffic, your perception of that average will be exaggerated.

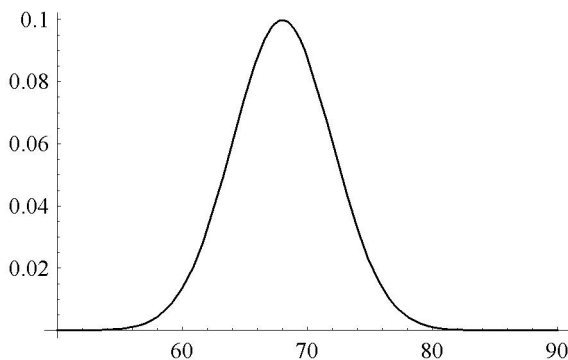
Perhaps most of this agrees with your intuition up to this point. However, it may surprise you to hear that this effect is more pronounced near the average speed than far below the average. In fact, if you are driving very slowly, your perception should be pretty accurate.

Suppose that the speeds of vehicles on a highway are normally distributed with mean 68 mph and standard deviation 4 mph (studies have shown that these assumptions are reasonable; see, for example [3, exhibits 9.1, 9.2, and 9.4]). Hence, the probability density function (pdf) is

$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-68}{4}\right)^2},$$

which has the graph given in Figure 1.

Suppose further that you are traveling at a speed of 65 mph. What is the distribution of speeds that you observe? We will assume that you accurately observe the speed of cars as they either pass you, or are passed by you; and that these are the only cars that you observe. The frequency with which you observe cars driving at speed  $x$  is thus in proportion to  $|x - 65|$ —the difference between the respective speeds. Hence, the

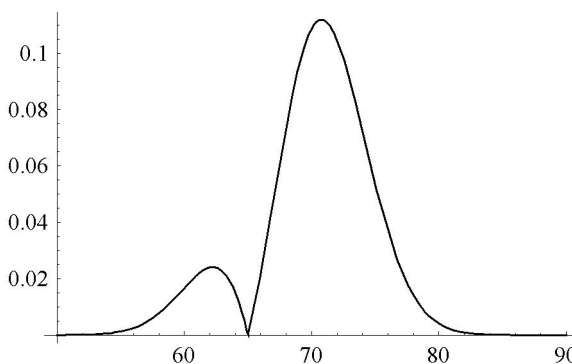


**Figure 1.** Assumed distribution of speeds on a highway.

distribution of speeds that you observe, after normalization, is

$$g(x) = \frac{|x - 65|f(x)}{\int_{-\infty}^{\infty} |t - 65|f(t) dt},$$

which has the graph given in Figure 2.



**Figure 2.** Distribution of observed speeds at 65 mph.

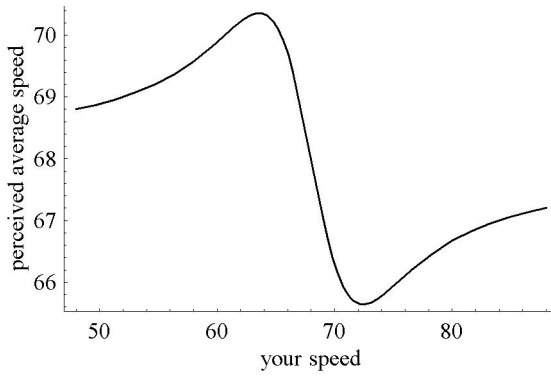
Notice that in our observations we have greatly reduced those cars going close to our speed, and proportionately increased the number of “speed demons” and “slow pokes.” The mean for this distribution,  $\mu_g$ , is approximately 70.16 mph. This exceeds the true average by over 2 mph.

Now we may wonder about the nature and size of this effect at other speeds. Suppose you are traveling at speed  $s$ ; let  $A(s)$  represent the perceived average speed of the other vehicles. Then

$$A(s) = \int_{-\infty}^{\infty} x \cdot \frac{|x - s|f(x)}{\int_{-\infty}^{\infty} |t - s|f(t) dt} dx.$$

The graph of  $A(s)$ , shown in Figure 3, represents the perceived average speed as a function of your speed.

Notice that this effect is most pronounced around 64 or 65 mph. If you are driving at exactly 68 mph, then your perception is accurate. If you are driving very slowly, your



**Figure 3.** Perceived average speed vs. your speed.

perception is only slightly exaggerated; in fact,  $y = 68$  is a horizontal asymptote for the graph. When you are driving faster than the average, the effect is reversed. Also, notice that if you are traveling at a speed near the actual average speed, then as you increase your speed, it seems as if the average speed of the other vehicles is decreasing. Conversely, as you decrease your speed, it seems as if the average speed of the other vehicles increases. In effect, traveling near the average speed here, your accelerator and brake appear more “touchy” than they actually are.

Let’s turn our attention to an attempt to prove some of these observed effects for more general distributions. Our first result says that if the actual distribution of speeds is symmetric (and thus  $\mu$  is the point of symmetry), and if you are traveling at the average speed, then your perception of the average speed of traffic will be accurate.

**Theorem 1.** *Let  $f$  be the probability density function of the speeds of vehicles on a highway, let  $\mu$  be the average value of  $f$  (assuming the existence of  $\mu$  is reasonable for realistic models), and let  $A(s)$  be the perceived average speed of vehicles for a driver traveling at speed  $s$ . If  $f$  is symmetric, then  $A(\mu) = \mu$ .*

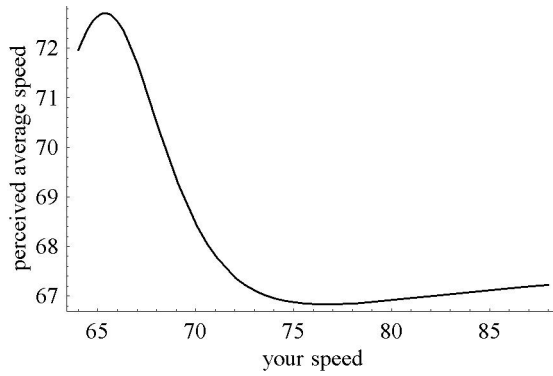
*Proof.* Since  $f$  is symmetric (about  $\mu$ ), then so is  $|x - \mu|f(x)$ . Hence,

$$\varphi(x) = \frac{|x - \mu|f(x)}{\int_{-\infty}^{\infty} |t - \mu|f(t) dt},$$

the probability distribution in question, is as well. Thus,

$$A(\mu) = \mu_{\varphi} = \mu. \quad \blacksquare$$

This result does not hold for non-symmetric distributions in general. Consider the shifted exponential distribution, also with mean 68 mph and standard deviation 4 mph, whose pdf is  $f(x) = (1/4)e^{-(x-64)/4}$ ,  $x \geq 64$ . When you travel at the actual average speed of 68 mph, you observe an average speed of approximately 70.56 mph. That is,  $A(\mu) > \mu$ . This means that although you are traveling at the average speed, your perception tells you that the average speed is higher, and you mistakenly believe that you are traveling slower than average. There is, however, a “fixed point”  $s_f$  of  $A$ , that is, a point where  $A(s_f) = s_f$ . This is the speed at which you perceive that you are going the average speed (here approximately 69.2 mph). This perception is also mistaken, for you would actually be going faster than the true mean speed. The graph of  $A(s)$  appears in Figure 4.



**Figure 4.** Perceived average speed vs. your speed, shifted exponential assumption.

The next result asserts an interesting property of the derivative of  $A(s)$ , a property which holds whether or not the distribution is symmetric. Consider Figure 3, and the fixed point 68. The slope of the curve is obviously negative, and a close inspection of the graph could lead the reader to suspect a derivative of  $-1$ . The same is true with the fixed point of 69.2 in Figure 4. This is startling because perfect perception would mean  $A(s) = \mu$  and  $A'(s) = 0$  for all  $s$ .

**Theorem 2.** *Let  $f$  be the probability density function of the speeds of vehicles on a highway, and let  $A(s)$  be the average speed of vehicles observed by a driver traveling at speed  $s$ . If  $s_f$  is a fixed point of  $A$ , then  $A'(s_f) = -1$ .*

*Proof.* Note that

$$\begin{aligned} A(s) &= \frac{\int_{-\infty}^{\infty} x|x-s|f(x) dx}{\int_{-\infty}^{\infty} |x-s|f(x) dx} \\ &= \frac{\int_{-\infty}^s x(s-x)f(x) dx + \int_s^{\infty} x(x-s)f(x) dx}{\int_{-\infty}^s (s-x)f(x) dx + \int_s^{\infty} (x-s)f(x) dx}. \end{aligned}$$

Then, using the quotient rule and Leibniz' rule, and letting bot and top denote the denominator and numerator of the previous fraction, respectively, we find that

$$\begin{aligned} A'(s) &= \frac{\text{bot} \cdot \left( \int_{-\infty}^s xf(x) dx + \int_s^{\infty} -xf(x) dx \right)}{\text{bot}^2} \\ &\quad - \frac{\text{top} \cdot \left( \int_{-\infty}^s f(x) dx + \int_s^{\infty} -f(x) dx \right)}{\text{bot}^2}. \end{aligned}$$

However, if we wish to evaluate this at the fixed point  $s_f$ , then we set  $\text{top} = s_f \cdot \text{bot}$ , and obtain

$$\begin{aligned} A'(s_f) &= \frac{\text{bot} \cdot \left( \int_{-\infty}^{s_f} xf(x) dx + \int_{s_f}^{\infty} -xf(x) dx \right)}{\text{bot}^2} \\ &\quad - \frac{s_f \cdot \text{bot} \cdot \left( \int_{-\infty}^{s_f} f(x) dx + \int_{s_f}^{\infty} -f(x) dx \right)}{\text{bot}^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{\int_{-\infty}^{s_f} x f(x) dx + \int_{s_f}^{\infty} -x f(x) dx - s_f \int_{-\infty}^{s_f} f(x) dx + s_f \int_{s_f}^{\infty} f(x) dx}{\int_{-\infty}^{s_f} (s_f - x) f(x) dx + \int_{s_f}^{\infty} (x - s_f) f(x) dx} \\
&= \frac{\int_{-\infty}^{s_f} (x - s_f) f(x) dx + \int_{s_f}^{\infty} (s_f - x) f(x) dx}{\int_{-\infty}^{s_f} (s_f - x) f(x) dx + \int_{s_f}^{\infty} (x - s_f) f(x) dx} \\
&= -1.
\end{aligned}$$

Observe that the difference between your speed and the observed average speed of other vehicles,  $s - A(s)$ , has derivative 2 at a fixed point of  $A$ . If you are in traffic that is heavy enough that observed differences may be noticed relatively quickly, and if you are driving at the speed that you perceive to be the average, then the accelerator and brake of your vehicle appear to have twice the effect that they actually have; that is, they appear twice as sensitive as normal. This assumes that you are judging the effect of the accelerator and brake by the change in  $s - A(s)$ , instead of by the change in  $s$ , which is a reasonable assumption when one is watching other vehicles and not objects at the side of the road. We are also assuming that, even though traffic is heavy, it is still moving freely, so that our assumptions about speeds of vehicles still hold. Perhaps this is one reason why driving in heavy traffic can be so nerve-wracking. Under the circumstances described, the relative changes in our motion due to braking and accelerating appear amplified by a factor of 2!

Figures 3 and 4 (and viewing similar models) suggest that this amplification factor is maximized at the mean (fixed point) for symmetric distributions, but not necessarily at the mean in the case of non-symmetric distributions. For instance, in the shifted exponential distribution given above, the maximum effect is greater than 2 and occurs at a point that is less than both the mean and the fixed point. We leave this problem open for further investigation.

Others too have observed questions of distorted highway perceptions; see [1] and [4]. See also [2] in a previous issue of this journal for an interesting discussion of how to determine the actual average speed of the cars on the highway. For the interested reader, *Mathematica* files exploring this topic are available at <http://www.uu.edu/personal/bdawson/highwayrelativity.html>.

So the next time you're on the road, although you may feel that traffic is getting the best of you, remember that perception is not always reality!

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